ANTENNA LABORATORY
Technical Report No. 58

ON INCREASING THE EFFECTIVE APERTURE OF ANTENNAS BY DATA PROCESSING

by
Robert H. MacPhie

Contract No. AF33(657)-8460
Project No. 6278, Task No. 40572

JULY 1962

Sponsored by
AERONAUTICAL SYSTEMS DIVISION
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

ELECTRICAL ENGINEERING RESEARCH LABORATORY
ENGINEERING EXPERIMENT STATION
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Antenna Laboratory

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ILLUSTRATIONS

1. INTRODUCTION

In recent years the concept of an antenna as a filter of spatial frequencies has come to play a central role in antenna theory in general and in Radio Astronomy in particular[1, 2, 3]. Given an antenna of finite size, its output as it scans the sky with its more or less narrow main beam and smaller sidelobes will be a smoothed version of the true source distribution of the sky.

This smoothing property is possessed by all finite antennas. In particular, if the antenna is linear (e.g., an array whose elements lie on a straight line in space), there is an exact one dimensional Fourier transform relation between the aperture weighting distribution $a(x)$, where x is the aperture coordinate in meters, and the field strength pattern $A(u)$, where $u = \beta \sin \theta$; β is the phase constant in radians per meter and θ is the polar radiation angle. Consequently $a(x)$ is the spatial frequency spectrum of $A(u)$, and is non-zero only over the finite interval $-L/2 \leq x \leq L/2$ where L is the length of the antenna in meters. This interval, centered at $x = 0$, is the spatial frequency bandwidth of the antenna and indicates that the antenna acts as a low-pass filter, i.e., a smoothing device.

A similar situation obtains in the case of the planar antenna[4]. The aperture coordinates x and y correspond to pattern coordinates $u = \beta \sin\theta \sin\phi$ and $v = \beta \sin \theta \cos \varphi$ and a two dimensional Fourier transform relation exists between the aperture weighting function and the field strength pattern. Most of the spatial frequency literature is concerned with either the linear or planar types of antennas, chiefly because of the above mentioned Fourier relationships which permit of exact quantitative analysis of the problem of spatial filtering. Consequently, this report considers only the linear type of antenna but the results can be quite easily extended to the planar case.

Now since the linear antenna is a low-pass filter of the ideal kind with a response to spatial frequencies lying outside the interval $-L/2 \leq x \leq L/2$ which is identically zero, it has been believed that no information about these higher frequency components is available at the antenna terminals. In theory at least, this is not so. If the sources which excite the receiving antenna are remote and radiate a finite amount of power in the direction of the

antenna, then by the Paley-Wiener Theorem[5] it can be shown that the spatial frequency spectrum of such a source distribution is a band-limited analytic function. Only the part lying in the interval $-L/2 \leq x \leq L/2$ is passed by the antenna. But this part is not independent of the other due to the analytic nature of the total spectrum. Since this function can be measured for the lower frequencies its values at the higher frequencies could be obtained by analytic continuation. Indeed if the function could be measured with arbitrarily high accuracy then an arbitrarily large extrapolation could be made. In effect, one would be calculating the higher spatial frequency components of the source distribution which would excite the antenna aperture if it were larger. The result of course would be an increase in the effective aperture and resolution of the antenna.

This problem is very similar to that of prediction in the theory of communication which has been thoroughly investigated by Wiener[6]. More recently Ville[7], in a short paper, considered the problem of speech prediction and by assuming that the frequency spectrum of the human voice had a compact support* arrived at the startling conclusion that by sampling an individual's speech over a finite period of time one could predict all future utterances of that person. However, it is an open question whether speech is a function let alone an analytic one. In addition, Ville showed that for his method of extrapolation the solution is very sensitive to small errors in the measured data and only a small extrapolation is practical. A paper by H. Wolter[8] on the analytic continuation of optical images has also shown that very little extension of the size of the image is possible due to measurement inaccuracies. Finally, two very recent papers by D. Slepian, H. J. Landau, and H. O. Pollak[9,10], which unfortunately were unknown to the author when the work of this report was being done, deal with the problem of taking finite samples of band limited signals and an elegant method of extrapolation is indicated but no quantitative results are given.

Other than in a short note by Y. T. Lo[11], there seems to be no work of a quantitative nature on the problem of extrapolation as applied specifically to linear antennas. The purpose of this report was to investigate this possibility. In particular an extrapolation of the time-averaged power spectrum of spatial frequencies was considered. The Fourier transform of

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*The term support used in distribution theory, means the complement of the open set over which a distribution (the voice's spectrum in this case) is zero.

this function is the temperature brightness distribution of the sky and is of principal interest to radio astronomers[1,3,4,11].

Witnessing the ease of formulation and mathematical advantages of the least square method, as well as its success in the case of Wiener,[6] it was decided to do the following:

1. approximate the spatial frequency spectrum on the finite interval $|x| \leq L$ by a Fourier polynomial whose coefficients are to be determined; this replaces the exact Fourier transform by a Fourier series of say N terms,

2. by the method of least squares obtain a set of N normal equations for the unknown coefficients.

Solution of the normal equations of course gives a least square approximation to the spectrum on the interval $|x| \leq L$. However, if the period of the approximating function is increased to a value greater than 2L and at the same time the number N is increased, then it is reasonable to suppose that due to the <u>analytic and band-limited nature</u> of the function being approximated, the Fourier polynomial will not only produce a least square fit over the original interval but to a certain extent will continue to approximate the function outside the interval. The amount of effective extrapolation would depend on the degree of the Fourier polynomial and its period. For the special case when N is four times the aperture length L in wavelengths the coefficient matrix of the normal equations becomes an identity matrix. However this case, whose solution can be obtained by inspection, does not correspond to an increase in effective aperture. Rather, it leads to the <u>principal solution</u> for the aperture of length L.

Unfortunately the extrapolation results were quite disappointing. Whenever the number of terms of the approximating polynomial and its period were increased to a point where the solution of the normal equations would have meant an increase in effective aperture, the equations inevitably became ill-conditioned. The coefficient matrix would always have an inverse whose elements were very large and of alternating sign. As a consequence, small errors in the measured data resulted in much larger relative errors in the calculated Fourier coefficients. In the report an analysis of the effect of such errors is made and for a given accuracy of measurement the amount of

useful extrapolation which one can expect, on the average, is determined. In addition, a simple example of the increase in resolution of an aperture of one wavelength that is achieved by this data processing method is described.

Although this problem is not quite the same physically as that of supergaining an antenna the end results are very similar. Thus, although the supergain antenna usually has a very large reactive field associated with it and the present one does not, the performances of both are very sensitive to small errors in the physical parameters of the antennas themselves.* The lack of highly reactive fields in the present system might be thought of as being replaced by the "highly reactive" coefficient matrix which must be used to process the measurement data. Since this matrix can be specified to an arbitrary number of decimal places one might think that on a suitably large digital computer the accuracy problem could be obviated. However, one still must make some physical measurements which lead to the elements on the right hand side of the set of normal equations. In general these elements cannot be specified to more than say four decimal places and hence the process is still strictly limited by the errors in the physical parameters of the system just as in the supergain case.

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*For a demonstration of the sensitivity of supergain antennas to small errors see E. C. Jordan, "Electromagnetic Waves and Radiating Systems," Prentice-Hall, 1950, pp. 445-450.

2. THE ANTENNA AS A FILTER OF SPATIAL FREQUENCIES

As was mentioned in the introduction, we shall, for simplicity, consider only the linear antenna by which is meant an antenna whose elements are located on a straight line in space. In this section the various Fourier transform relations between the far field source distribution, the antenna pattern and their corresponding spatial frequency spectra will be formally presented.

Now just as in communication theory where a specific time signal is described exactly by some complex voltage spectrum and a random signal is usually described statistically by some real average power spectrum (in which all phase information is lost), it is also true in linear antenna theory that the exact description is in terms of the field strength or voltage spectrum, while the time-averaged statistical description is more convenient in terms of power. For brevity these two types of spectra will be called voltage and power spectra and will receive separate treatment.

2.1 The Voltage Description

Let us consider an antenna of length L meters as is shown in Figure 1. If there is a remote point source of unit field strength in the direction θ, then it is well known that except for a constant factor the terminal voltage due to the source is

$$v_1(\theta_o, t) = \mathrm{Re} \int_{-L/2}^{L/2} a(x)e^{j\beta x \sin\theta_o} \; e^{-j\beta x \sin\theta} dx e^{j\omega_o t} \qquad (1)$$

Re indicates "The real part of..."

x = distance measured in meters along the aperture,

$a(x)$ = the aperture weighting function and may be complex,

β = propagation constant in radians per meter,

$\beta\sin\theta_o$ = progressive phase shift across the aperture in radians per meter which causes the main lobe of the antenna pattern to be pointed in the θ_o direction,

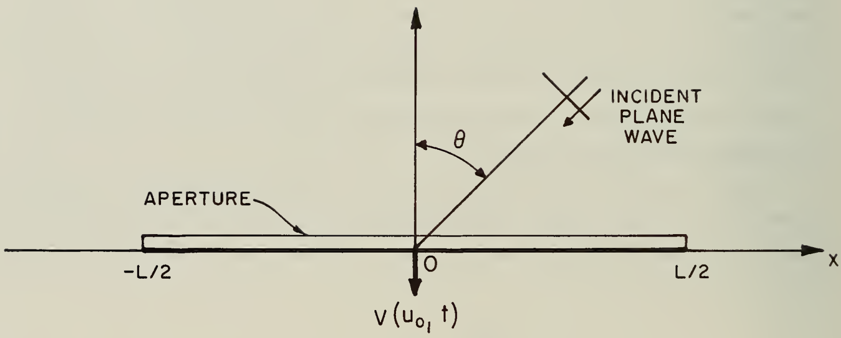

Figure 1. Coordinate System of a Linear Antenna

ω_o = angular frequency of the signal in radians per second,

t = time in seconds,

j = $+\sqrt{-1}$.

If one lets

$$u = \beta \sin \theta \tag{2}$$

Equation (1) can be rewritten as

$$v(u_o, t) = \text{Re} \int_{-L/2}^{L/2} a(x) e^{jxu_o} e^{-jxu} dx e^{j\omega_o t} \tag{3}$$

Equation (2) is illustrated graphically in Figure 2. If β is thought of as a propagation vector $\vec{\beta}$, which indicates the direction of arrival as well as the frequency of the signal, then u is its scalar projection on a line parallel to the x axis and varies from $-\beta$ to β for $\theta = -\pi/2$ and $\pi/2$ respectively. Note also that the sources at θ and $\pi-\theta$ have the same projection and hence are indistinguishable.

Returning to Equation (3) we see that since the aperture is finite, $a(x) = 0$ for $|x| > L/2$ and we can write

$$v(u_o, t) = \text{Re}\, A(u-u_o)\, e^{j\omega_o t} \tag{4}$$

where the field strength pattern

$$A(u) = \int_{-\infty}^{\infty} a(x) e^{-jux} dx = \int_{-L/2}^{L/2} a(x) e^{-jux} dx \tag{5}$$

is the Fourier transform of a(x), the aperture weighting function.

Mathematically, the location of the point source in the u_1 domain can be specified by the Dirac delta function, $\delta(u_1-u)$ and the terminal voltage of the antenna can be written as

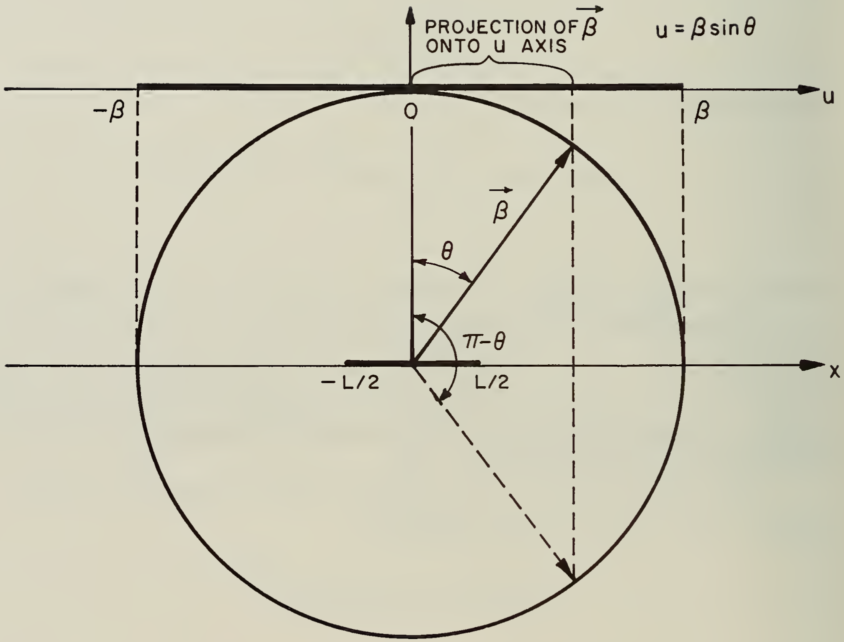

Figure 2. The Relation between u and θ

$$v(u_o,t) = \text{Re} \int_{\infty}^{\infty} \delta(u_1-u)\ A(u_1-u_o)\ du_1 e^{j\omega_o t} \tag{6}$$

$$\text{Re}\ \delta(u_o-u) * \check{A}(u_o) e^{j\omega_o t} \tag{7}$$

where * is the symbol for convolution. Here we have defined

$$\check{A}(u) = A(-u) \tag{8}$$

Thus the point source or delta function response of the antenna is the reverse of its field strength pattern. In the majority of practical cases the pattern is an even function and the point source response and the pattern are identical.

Now in general there is not one source but a distribution of sources. The instantaneous field strength at the phase center of the antenna due to a plane wave from the source in the u direction at time t can be written as

$$e(u,t) = \text{Re}\, \mathcal{E}(u,t)\ e^{j\omega_o t} \tag{9}$$

where the complex phasor $\mathcal{E}(u,t)$ represents the amplitude and phase of the narrow band envelope of the carrier at frequency ω_o.* The total output voltage of the antenna is the integral of all such plane waves weighted by the pattern function $\check{A}(u_o-u)$.

$$v(u_o,t) = \text{Re} V(u_o,t)\ e^{j\omega_o t} \tag{10}$$

$$= \text{Re} \int_{-\beta}^{\beta} \mathcal{E}\ (u,t) . \check{A}(u_o-u)\ du\ e^{j\omega_o t} \tag{11}$$

$$= \text{Re}\, \mathcal{E}\,(u_o,t) * \check{A}(u_o)\ e^{j\omega_o t} \tag{12}$$

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*It has been tacitly assumed that there is an RF filter which allows only a narrow band of frequencies centered at ω_o to exist at the antenna terminals.

The limits of $\pm\beta$ on the integral indicate that the sources are in the visible range.

Equation (12) shows that the complex amplitude $V(u_o,t)$ of the antenna's terminal voltage is equal to the convolution of the source distribution and the reverse of the antenna pattern. This function has an inverse Fourier transform, $v(x,t)$, which with the aid of the convolution theorem can be written as

$$v(x,t) = \epsilon(x,t)\,\check{a}(x) \qquad (13)$$

where $\epsilon(x,t)$ is the inverse Fourier transform of $\mathcal{E}(u,t)$ and $\check{a}(x)$ is the reverse of the aperture weighting function. Since $a(x)$ is identically zero for $|x| > L/2$ it is clear from Equation (13) that the voltage spectrum of the output is also zero in these regions. If the modulus of $a(x)$ is greater than zero for all values of x in the aperture then from Equation (13) we define

$$\epsilon_o(x,t) = \frac{v(x,t)}{\check{a}(x)}, \quad |x| \leq L/2 \qquad (14)$$

$$= 0, \quad |x| > L/2$$

whose Fourier transform is

$$\mathcal{E}_o(u_o,t) = \int_{-L/2}^{L/2} \frac{v(x,t)}{\check{a}(x)}\, e^{-ju_o x}\, dx \qquad (15)$$

This last function is known as the <u>principal solution.</u> It is a smoothed version of the true source distribution with no spatial frequencies greater in absolute value than L/2. However the frequency components which are present are identical to those of the true source distribution $\mathcal{E}(u_o,t)$. From Equation (13) it is clear that the weighting function $\check{a}(x)$, in general, distorts the source distribution's spectrum as "seen" at the antenna output in the form of $v(x,t)$. But, if the antenna aperture is uniformly weighted

with $\check{a}(x) = 1$ for $|x| \le L/2$, then $m(x,t) = \mathcal{E}(x,t)$ in that interval and the output as the antenna scans will be the principal solution $\mathcal{E}_o(u_o,t)$. If a(x) is not uniform, the principal solution can still be recovered by measuring $V(u_o,t)$ and using Equations (13) and (15).

Figure 3 is a table of the various pairs of Fourier transforms which have been derived above. The RF carrier factor has been suppressed.

2.2 The Power Description

In many applications it is not the instantaneous signal from a remote source that is of interest but simply its average power. The fields of radio astronomy, radiometry, radio direction finding, and radar are all more or less concerned with measuring the average power radiated or reflected by the remote sources.

A diagram of the conventional system for measuring such power is shown in Figure 4. The terminal voltage of the antenna is fed into a square law device whose output is

$$v^2(u_o,t) = [\mathrm{Re}\,\mathcal{E}(u_o,t) * \check{A}(u_o)\, e^{j\omega_o t}]^2 \tag{16}$$

$$\begin{aligned} &= 1/2\ \mathrm{Re}\,[\mathcal{E}(u_o,t) * \check{A}(u_o)]^2\, e^{j2\omega_o t} \\ &+ 1/2\ \mathrm{Re}\ \mathcal{E}(u_o,t) * \check{A}(u_o) \cdot \mathcal{E}^*(u_o,t) * \check{A}^*(u_o) \end{aligned} \tag{17}$$

The first term on the right side of Equation (17) is the double frequency component and has a zero average value. The second term is the low frequency component which along contributes to the low pass filter output. Thus, if we assume that the filter response is that of an ideal averager, then the system output after a suitably long integration time approaches

$$R(u_o) = E\left\{\frac{1}{2}\ \mathrm{Re}\ \mathcal{E}(u_o,t) * \check{A}(u_o) \cdot \mathcal{E}^*(u_o,t) * \check{A}^*(u_o)\right\} \tag{18}$$

where the superscript * indicates the complex conjugate and $E\left\{f(u_o)\right\}$ is the statistical expectation of $f(u_o)$. Here it is assumed that time and statistical averaging are equivalent (the sources are ergodic). Equation (18) can be rewritten as

DESCRIPTION	VOLTAGE TRANSFORM PAIRS u - radians/meter	 x - meters	DESCRIPTION
FIELD STRENGTH PATTERN	$A(u)$	$a(x)$ $(a(x) = 0 \text{ for } \vert x \vert > \frac{L}{2})$	APERTURE WEIGHTING FUNCTION
POINT SOURCE RESPONSE	$\check{A}(u) = A(-u)$	$\check{a}(x) = a(-x)$	
SOURCE DISTRIBUTION	$\mathcal{E}(u,t)$ $(\mathcal{E}(u,t) = 0 \text{ for } \vert u \vert > \beta)$	$\epsilon(x,t)$	FIELD ON X AXIS DUE TO SOURCES
TERMINAL VOLTAGE PHASOR	$V(u,t) = \mathcal{E}(u,t) * \check{A}(u)$	$v(x,t) = \epsilon(x,t)\, \check{a}(x)$	
PRINCIPAL SOLUTION	$\mathcal{E}_o(x,t)$	$\epsilon_o(x,t) = \frac{v(x,t)}{\check{a}(x)}, \vert x \vert \leq \frac{L}{2}$ $= 0, \quad \vert x \vert > \frac{L}{2}$	APERTURE FIELD DUE TO SOURCES

Figure 3. Fourier Transform Pairs for the Voltage Description of the Antenna as a Filter of Spatial Frequencies

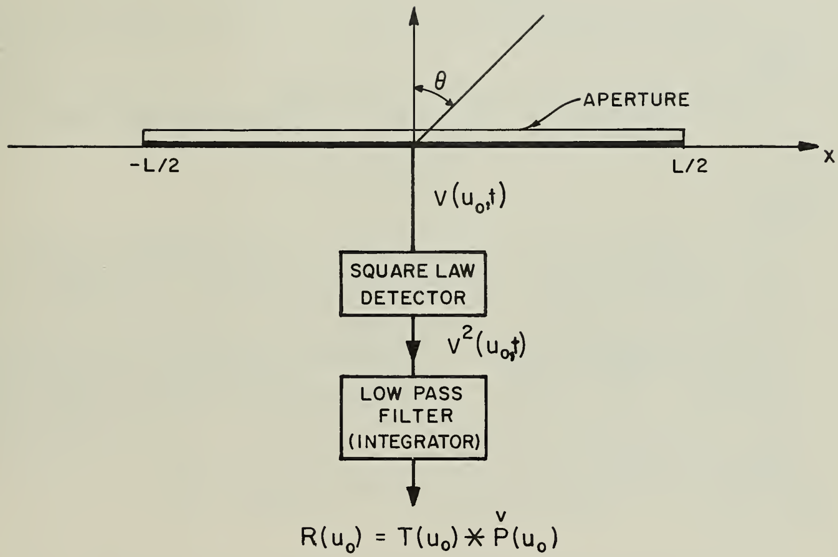

Figure 4. Conventional Antenna System Employing Square Law Detection and Low Pass Filtering

$$R(u_o) = \frac{1}{2} \text{ Re} \int_{-\beta}^{\beta} \int_{-\beta}^{\beta} E \left\{ \mathcal{E}(u_1,t)\, \mathcal{E}^*(u_2,t) \right\} \check{A}(u_o-u_1)\, \check{A}^*(u_o-u_2)\, du_1\, du_2 \tag{19}$$

In a wide range of applications the following statistical properties of the phasor $\mathcal{E}(u,t)$ obtain:

If

$$\mathcal{E}(u,t) = \mathcal{E}_1(u,t) + j\,\mathcal{E}_2(u,t) \tag{20}$$

then

$$E\left\{\mathcal{E}_1(u,t)\right\} = E\left\{\mathcal{E}_2(u,t)\right\} = E\left\{\mathcal{E}_1(u,t)\,\mathcal{E}_2(u,t)\right\} = 0 \tag{21}$$

and

$$2E\left\{\mathcal{E}_1^{\,2}(u,t)\right\} = 2E\left\{\mathcal{E}_2^{\,2}(u,t)\right\} = E\left\{|\mathcal{E}(u,t)|^2\right\} = T(u) \tag{22}$$

The real and imaginary parts of $\mathcal{E}(u,t)$ have zero means and are statistically independent. Their variances are equal to 1/2 $T(u)$, where the real number $T(u)$ is the variance of $\mathcal{E}(u,t)$ and is a measure of the <u>temperature brightness of the source</u> in the direction u. Usually the source in one direction is time-wise independent of the sources in all other directions and so one can write

$$E\left\{\mathcal{E}(u_1,t)\,\mathcal{E}^*(u_2,t)\right\} = T(u_1)\,\delta\,(u_1-u_2) \tag{23}$$

Substituting this result back into Equation (19) gives

$$R(u_o) = \frac{1}{2} \text{ Re} \int_{-\beta}^{\beta} T(u_1)\, A\,(u_o-u_1)\, A^*(u_o-u_1)\, du_1 \tag{24}$$

$$= \frac{1}{2}\, T(u_o) * |A(u_o)|^2 \tag{25}$$

or if we let

$$\frac{1}{2} | \check{A}(u_o) |^2 = \check{P}(u_o) \tag{26}$$

then

$$R(u_o) = T(u_o) * \check{P}(u_o) \tag{27}$$

is the time-averaged response of the antenna as a function of the scan "angle" u_o.

Again we can take the inverse Fourier transform of this result and by the convolution theorem we have

$$r(x) = t(x)\, \check{p}(x) \tag{28}$$

where $r(x)$, $t(x)$ and $\check{p}(x)$ are the inverse Fourier transforms of $R(u_o)$, $T(u_o)$ and $\check{P}(u_o)$ respectively. In particular, $\check{p}(x)$ is the transform of $\frac{1}{2} | \check{A}(u_o) |^2$ (Equation 26), and again by invoking the convolution theorem one obtains the following relation between the power spectrum of spatial frequencies $p(x)$ and the voltage spectrum $a(x)$ for the antenna.

$$\check{p}(x) = \frac{1}{2} \check{a}(x) * \check{a}^*(-x) = \frac{1}{2} a(-x) * a^*(x) \tag{29}$$

Since $a(x) \equiv 0$ for $| x | > \frac{L}{2}$ it follows from the properties of the convolution operation that $p(x) \equiv 0$ for $| x | > L$. For example, if $a(x)$ were uniform in the interval $| x | \leq \frac{L}{2}$ then $\check{p}(x)$ would be triangular in the interval $| x | \leq L$. Thus the band-width of power spatial frequencies is twice that for the corresponding voltage frequencies. This doubling effect also occurs in the time domain.[12]

Since $p(x) \equiv 0$ for $| x | > L$ we know that $R(u_o)$ contains no spatial frequencies for $| x | > L$ and those for $| x | \leq L$ will be related to the spatial

frequencies of the brightness distribution $T(u)$ by Equation (28). Just as in the voltage case the principal solution $T_o(u)$ can be obtained. Thus we define

$$t_o(x) = \frac{r(x)}{\check{p}(x)}, \qquad |x| \leq L$$
$$= 0 \quad , \qquad |x| > L \tag{30}$$

and

$$T_o(u) = \int_{-L}^{L} \frac{r(x)}{\check{p}(x)} e^{-jux} \, dx \tag{31}$$

Indeed the term principal solution was first applied to this type of power distribution by Bracewell and Roberts[1] rather than to the instantaneous voltage distribution of Equation (15). $T_o(u)$ is a smoothed version of $T(u)$; its frequency components for $|x| \leq L$ are identical to those of $T(u)$ and are equal to zero for $|x| > L$.

It would be convenient if the antenna power spectrum $\check{p}(x)$ were uniform. Then $r(x)$ would be equal to $t_o(x)$ (except possibly for a constant factor) and the antenna output $R(u_o)$ would be the desired principal solution. Although in the voltage case this can be accomplished with little difficulty by letting $a(x) = 1$ for $|x| \leq \frac{L}{2}$, it is impossible in the power case when a single antenna and a square-law detection are used. No aperture function $a(x)$ exists which leads to a uniform power spectrum $\check{p}(x) = 1$, through the convolution operation of Equation (29). However, if the terminal voltages of two antennas are cross-correlated it can be shown[13] that the power pattern that results is

$$P_2(u) = \frac{1}{2} \operatorname{Re} A(u) \, B^*(u) \, e^{j\ell u} \tag{32}$$

where $A(u)$ is the pattern of the first antenna,

$B(u)$ is the pattern of the second

and $e^{j\ell u}$ is an interferometer-type pattern resulting from the spacing of ℓ

meters between the antennas. If we let

$$B_\ell(u) = B(u)\ e^{-j\ell u} \tag{33}$$

Equation (32) can be written as

$$P_2(u) = \frac{1}{2} \text{Re } A(u)\ B_\ell^*(u)$$

or

$$\check{P}_2(u) = \frac{1}{2} \text{Re } \check{A}(u)\ \check{B}_\ell^*(u) \tag{34}$$

The associated spatial frequency spectrum is

$$\check{p}_2(x) = \frac{1}{4}\left[\check{a}(x) * b_\ell^*(x) + a^*(x) * \check{b}_\ell(x)\right] \tag{35}$$

Now if the correlation system is in the form of a Compound Interferometer,[14] then the spectrum $\check{p}_2(x)$ will be uniform and the system output will be the principal solution $T_o(u_o)$. A diagram of this system is shown in Figure 5. The two antennas are a uniformly weighted aperture of length $\frac{L}{2}$ meters adjacent to a simple interferometer also of length $\frac{L}{2}$ meters. Thus we let

$$\begin{aligned} a(x) &= 2, \qquad -\frac{L}{2} \leq x < 0 \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{36}$$

$$b_\ell(x) = 2\delta(x) + 2\delta(x-\frac{L}{2}) \tag{37}$$

Substituting these functions into Equation (35) gives

$$\begin{aligned} \check{p}_2(x) &= 1 \qquad |x| \leq L \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{38}$$

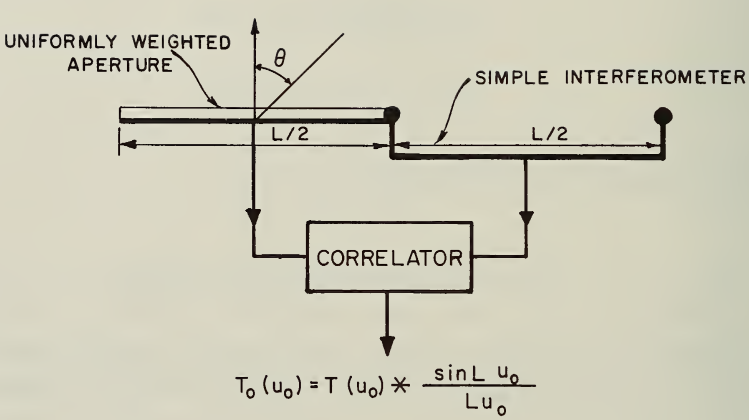

Figure 5. Compound Interferometer System

and

$$P_2(u) = L \frac{\sin Lu}{Lu} \tag{39}$$

the correlator output is

$$R(u_o) = T_o(u_o) = T(u_o) * L \frac{\sin Lu_o}{Lu_o} \tag{40}$$

A table of the various Fourier transform pairs for the power description is given in Figure 6. Like all Fourier transform relations these results are <u>linear</u> (in average power in this case). This very fortunate property is due to the assumption that sources in different directions are statistically independent in time.

Because of the rather startling fact that an antenna is a <u>perfect</u> low-pass filter of both the voltage and power spatial frequencies of a source distribution it has generally been concluded that the principal solution is the most one can hope for and that no information about the higher spatial frequencies is available at the terminals of the antenna. In the next chapter it will be shown that this is not the case.

POWER TRANSFORM PAIRS

DESCRIPTION	u - Radians/meter	x - meters	DESCRIPTION
ANTENNA POWER PATTERN	$P(u) = \frac{1}{2} A(u)\ A^*(u)$	$p(x) = \frac{1}{2} a(x) * a^*(-x)$ $(p(x) = 0 \text{ for } \lvert x \rvert > L)$	ANTENNA POWER SPECTRUM
POINT SOURCE RESPONSE	$\check{P}(u) = P(-u)$	$\check{p}(x) = p(-x)$	
SOURCE DISTRIBUTION	$T(u)$ $(T(u) = 0 \text{ for } \lvert u \rvert > \beta)$	$t(x)$	SPECTRUM OF T(u)
ANTENNA OUTPUT POWER	$R(u_o) = T(u) * \check{P}(u)$	$r(x) = t(x)\ \check{p}(x)$	
CORRELATION ANTENNA PATTERN	$P_2(u) = \frac{1}{2}\mathrm{Re}\ A(u)\ B_\ell^*(u)$	$p_2(x) = \frac{1}{4}\check{a}(x) * b_\ell^*(x)$ $+ \frac{1}{4} a^*(x) * \check{b}_\ell(x)$	
PRINCIPAL SOLUTION	$T_o(u)$	$t_o(x) = \frac{r(x)}{\check{p}(x)}, \lvert x \rvert \leq L$ $= 0, \lvert x \rvert > L$	LOW FREQUENCY SPECTRUM OF T(u)

Figure 6. Fourier Transform Pairs for the Power Description of the Antenna as a Filter of Spatial Frequencies

3. PROPERTIES OF THE TEMPERATURE DISTRIBUTION T(u) AND ITS SPATIAL FREQUENCY SPECTRUM t(x)

In this chapter mathematical models of the temperature distribution T(u) and of its Fourier transform t(x) will be derived from some general assumptions about the physical nature of the sources whose radio waves are intercepted by the receiving antenna. Indeed, in the remainder of this report we will be concerned only with the time-averaged functions T(u) and t(x) rather than with their instantaneous counterparts $\mathcal{E}(u,t)$ and $\epsilon(x,t)$. This is partly for convenience and partly for historical reasons since most of the early work dealt with the former functions.

3.1 The Temperature Distribution - T(u)

It will be recalled (Chapter 2.1) that the variable u is related to the physical direction θ by Equation (2)

$$u = \beta \sin \theta$$

Thus T(u) is the projection on a line parallel to the antenna axis of the actual brightness distribution $T_A(\theta)$ and hence is a distorted version of it. There is no difficulty, however, in correcting for this distortion. Given T(u), there are two values of θ corresponding to say $u = u_1$, $\theta_1' = \sin^{-1} u_1/\beta$ and $\theta_1'' = \pi - \theta_1'$ and hence

$$T_A(\theta_1') + T_A(\theta_1'') = T(u_1) \tag{41a}$$

If the antenna is located on a ground plane, then the source at θ and image at $\pi - \theta$ are always identical and we get a unique relation between a source in the upper half space and its projection, i.e.,

$$T_A(\theta) = T_A(\sin^{-1} \frac{u}{\beta}) = \frac{1}{2} T(u) \tag{41b}$$

Any mathematical model of a source distribution must necessarily be rather general and incomplete because if one knew exactly the properties of the temperature distribution it would not be necessary to ever build

devices to measure them. In a wide range of application the sources possess the following properties:

1. they are stationary (with respect to their time statistics),
2. they are remote from the receiving antenna and fixed in space for the duration of the time of observation,
3. a source in one direction is statistically independent (in the time domain) of a source in any other direction.

Indeed these three properties have been used to derive Equation (27) for the time-averaged response of the antenna. The function T(u) is a measure of the temperature brightness of the remote sources and is the expected value of the absolute square of the field phasor $\mathcal{E}(u,t)$

$$T(u) = E\left\{|\mathcal{E}(u,t)|^2\right\} \tag{42}$$

Sometimes it is written as, kT(u) where k is Boltzmann's constant and hence is a measure of the energy being radiated by the sources toward the antenna system. Consequently, T(u) is non-negative since it is assumed that the average energy flow from the sources cannot be negative. It has also been assumed that the sources are remote. This means that they are in the so-called visible range of the antenna which requires the angle θ to be real. As a result the sources all lie in the range for which $|u| \leq \beta$.* Outside this range the source distribution is identically zero. Indeed the word "distribution" is a better description of the sources in space than the word "function". This is because in some applications it is convenient to represent remote sources as "point" sources whose mathematical description is in terms of delta "functions" which of course are distributions. In summary the model of the temperature distribution T(u) has the following properties:

1. T(u) is non-negative and real,
2. $T(u) = 0$ for $|u| > \beta$, (compact support)

<u>3.2 The Spatial Frequency Spectrum - t(x)</u>

The inverse Fourier transform of T(u) is by definition

$$t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(u)\, e^{jux}\, du \tag{43}$$

---

*The source distribution T(u) has a compact support.

But since $T(u) = 0$ for $|u| > \beta$ we can write

$$t(x) = \frac{1}{2\pi} \int_{-\beta}^{\beta} T(u)\, e^{jux}\, du \tag{44}$$

This is the inverse Fourier transform of a real function and hence possesses complex symmetry, i.e., if

$$t(x) = t_1(x) + j\, t_2(x) \tag{45}$$

then

$$t_1(x) = t_1(-x), \quad t_2(x) = -t_2(-x) \tag{46}$$

or

$$t(x) = t^*(-x) \tag{47}$$

Physically, $t(x)$ is a function which depends on the field at one point x_1, with the field at another point x_2, where $x = x_1 - x_2$. In complex notation it is given by half of the expected value of the Hermitian product of the field phasors at the two points

$$t(x) = \frac{1}{2} E \left\{ \epsilon(x_1, t)\ \ \epsilon^*(x_2, t) \right\} \tag{48}$$

This function is therefore measurable, at least for some finite values of its argument. Since it is the Fourier transform of a distribution which is identically zero for $|u| > \beta$ (compact support), it follows from the Paley-Wiener theorem[5] that it is a band-limited analytic function and never becomes identically zero over any finite interval of x.*

However, $t(x)$ must satisfy the energy condition (Parseval's theorem)

$$\int_{-\infty}^{\infty} |t(x)|^2\, dx = \int_{-\beta}^{\beta} |T(u)|^2\, du \tag{49}$$

---

*By "band limited function" we mean one whose Fourier transform has a compact support.

and since it is assumed that the sources radiate a finite amount of energy towards the antenna the integral on the right exists. Consequently, in order that the improper integral of $|t(x)|^2$ converge we must have

$$\lim_{x \longrightarrow \infty} |t(x)|^2 = O[(\frac{1}{x})^{1+\epsilon}] \tag{50}$$

where $O(\frac{1}{x})$ means "of order at most $\frac{1}{x}$" and $\epsilon > 0$.

The function $t(x)$ is related to the complex degree of coherence[15] between the field at the two points x_1 and x_2 which is given by

$$\gamma_{12}(\tau) = \frac{E\left\{\epsilon(x_1,t)\,\epsilon^*(x_2,t-\tau)\right\}}{E\left\{|\epsilon(x_1,t)|^2\right\}^{1/2} E\left\{|\epsilon(x_2,t)|^2\right\}^{1/2}} \tag{51}$$

This function also depends on the time delay τ. When $\tau = 0$ the numerator reduces to $2t(x)$ and in the case of remote sources the denominator reduces to $2t(o)$. Hence we can write

$$\gamma_{12}(o) = \frac{t(x)}{t(o)} \tag{52}$$

which is the normalized spatial frequency spectrum and is valid when all sources are remote.

To summarize, the spatial frequency spectrum $t(x)$ has the following properties:

1. $t(x) = t^*(-x)$, (complex symmetry)
2. $t(x) = 1/2\, E\left\{\epsilon(x_1,t)\,\epsilon^*(x_2,t)\right\}$ with $x_1 - x_2 = x$, (measurability)
3. $t(x)$ is analytic,
4. $t(x)$ is band-limited,
5. $\lim_{|x| \longrightarrow \infty} |t(x)|^2 = O[(\frac{1}{x})^{1+\epsilon}]$, $\epsilon > 0$ (finite energy)

In the next part it will be shown that $t(x)$, or rather a finite section of it, can be measured directly by means of a correlation interferometer.

3.3 Direct Measurement of the Spatial Frequency Spectrum

In Chapter 2.2 it was shown that by analyzing the output of an antenna of length L one could deduce the spatial frequency spectrum of the source distribution that was passed by the antenna. This spectrum, $t_o(x)$, is a truncated version of the true spectrum, being equal to $t(x)$ for $|x| \leq L$ and identically zero for $|x| > L$. It is also possible to measure directly $t_o(x)$ by means of one of the simplest of antennas.

Consider a two-element interferometer in which the terminal voltages of the two isotropic elements are cross-correlated as is shown in Figure 7. The phase shifter in the feed line from the right-hand element can be used for scanning but if it is set at zero and if the element spacing is x meters, then the system output due to a source distribution $T(u)$ is

$$t_1(x) = \frac{1}{2\pi} \int_{-\beta}^{\beta} T(u) \cos ux \, du \qquad (53)$$

The limits $\pm \beta$ on the integral are due to our assuming that all sources are in the visible range.

If the phase shifter is set at $\pi/2$ radians, the output is

$$t_2(x) = \frac{1}{2\pi} \int_{-\beta}^{\beta} T(u) \sin ux \, du \qquad (54)$$

thus we can write

$$t(x) = t_1(x) + j\, t_2(x) = \frac{1}{2\pi} \int_{-\beta}^{\beta} T(u)\, e^{jux} \, du \qquad (55)$$

or

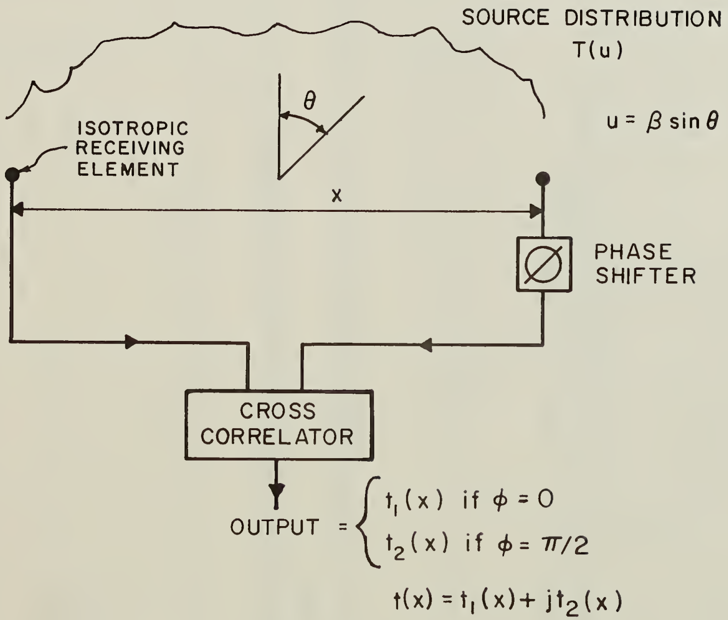

Figure 7. Cross-correlation Interferometer

$$t(x) = F^{-1}\left\{T(u)\right\} \tag{56}$$

the inverse Fourier transform of T(u).

In practice the interferometer element spacing x cannot exceed some maximum value, say L meters. Hence the function which we measure is

$$\begin{aligned} t_o(x) &= t(x) \qquad \text{for } |x| \leq L \\ &= 0 \qquad \text{for } |x| > L \end{aligned} \tag{57}$$

The value of $t_o(x)$ for negative x has been deduced from the complex symmetry property of $t_o(x)$, i.e., $t_o(-x) = t_o^*(x)$. Thus, by varying the element spacing of a correlation interferometer we can measure directly the complex spatial frequency power spectrum of the distance sources on the interval $|x| \leq L$.

3.4 Analytic Continuation

Perhaps the most important property of t(x) is its analyticity. It is well known that if an analytic function is specified exactly over any finite region its value at any point outside the region can be deduced by the process of analytic continuation[16]. Since t(x) is analytic for all values of x and its values for $|x| \leq L$ can be measured, it is theoretically possible to calculate t(x) for any x. This would yield the complete spatial frequency spectrum t(x) for $-\infty \leq x \leq \infty$ and the true temperature brightness distribution T(u) could be obtained by taking its Fourier transform.

Of course this assumes that t(x) can be measured exactly. For such an idealized situation it is interesting to note that the size of the measurement interval is of no importance. As long as t(x) is measured with no error an aperture of one wavelength is "equivalent" to one of a thousand wavelengths! However, error is unavoidable and in its presence the amount of meaningful extrapolation of t(x) will be shown, later in this report, to be very limited. In practice, therefore, the large aperture is still preferable to the small aperture.

The problem is to find a suitable method for performing the extrapolation. In Chapter 4 a method which takes into account the Fourier and band-limited aspects of t(x) will be described.

4. FOURIER POLYNOMIAL APPROXIMATION

Having obtained by measurement the real empirical curves $t_1(x)$ and $t_2(x)$ for $-L \leq x \leq L$, it is necessary to express them analytically by means of mathematical functions. Since $t(x) = t_1(x) + jt_2(x)$ is known to be the Fourier transform of the unknown source distribution $T(u)$, it is natural to approach the problem with the methods of Fourier analysis. We will use a Fourier polynomial to obtain a least square approximation to $t(x)$ on the interval $-L \leq x \leq L$. Then by increasing both the degree N and the period L_o of the polynomial we will attempt to increase the least square fit on the interval to the point where it continues to approximate the function $t(x)$ for points just outside the interval. This continuation is predicated on the fact that $t(x)$ is analytic.

4.1 Fourier Polynomial Approximation to t(x)

By definition we have

$$t(x) = \frac{1}{2\pi} \int_{-\beta}^{\beta} T(u)\, e^{jux}\, du \tag{58}$$

If the visible range $-\beta \leq u \leq \beta$ is divided into N equal regions*, the kth of which is centered at

$$u_k = (k + \frac{1}{2}) \frac{2\beta}{N}$$

then we can write

$$t(x) = \sum_{k=-N/2}^{N/2-1} \frac{1}{2\pi} \int_{k\frac{2\beta}{N}}^{(k+1)\frac{2\beta}{N}} T(u)\, e^{jux}\, du \tag{59}$$

---

*We have for convenience chosen N even.

Letting $u = v + u_k$ this becomes

$$t(x) = \sum_{k=-N/2}^{N/2-1} \frac{1}{2\pi} \int_{-\beta/N}^{\beta/N} T(v + u_k)\, e^{jvx}\, dv\, e^{ju_k x} \tag{60}$$

$$= \sum_{k=-N/2}^{N/2-1} T_k^{(N)}(x)\, e^{ju_k x} \tag{61}$$

where

$$T_k^{(N)}(x) = \frac{1}{2\pi} \int_{-\beta/N}^{\beta/N} T(v+u_k)\, e^{jvx}\, dv \tag{62}$$

If we expand the exponential in the integral as a power series in (jvx), the integral becomes

$$T_k^{(N)}(x) = \int_{-\beta/N}^{\beta/N} T[v+u_k] \left\{ 1 + jvx - \frac{(vx)^2}{2} + \ldots \right\} dv \tag{63}$$

The leading term is clearly $2\beta/N$ times the average value of $T[v + u_k]$ on the interval $|v| \leq \beta/N$ and if we assume that $T(u)$ is continuous, then

$$\lim_{N \to \infty} \frac{N}{2\beta} \int_{-\beta/N}^{\beta/N} T[v + u_k]\, e^{jvx}\, dv = \frac{N}{2\beta} \int_{-\beta/N}^{\beta/N} T[v + u_k]\, dv \tag{64}$$

$$= T(v + u_k)$$

Thus we let

$$t^{(N)}(x) = \frac{\beta}{\pi N} \sum_{k=-N/2}^{N/2-1} T_k^{(N)} e^{ju_k x} \tag{65}$$

which is a Fourier polynomial of N terms with a period L_o meters, given by

$$L_o = \frac{2\pi}{\beta/N} = N\lambda \tag{66}$$

where $\lambda = 2\pi/\beta$ is the wavelength of the signal in meters. The real, constant coefficient $T_k^{(N)}$ is the approximation to the average value of T(u) on the kth interval. The superscript N indicates the degree of the approximating polynomial. Figure 8a is a graph of a "typical" source distribution which has been partitioned into 10 sections (N = 10). In Figure 8b is shown a distribution of 10 delta functions the kth of which is located at $u_k = (k + \frac{1}{2})\frac{2\beta}{N}$ and has a magnitude $T^{(N)}k$ which is the approximation to the average value of T(u) on the kth interval. The distribution can be given mathematically by the formula

$$T^{(N)}(u) = \sum_{k=-N/2}^{N/2-1} T_k^{(N)} \delta \left\{ u - u_k \right\} \tag{67}$$

Except for the constant factor $\frac{2\beta}{N}$, the Fourier transform of $T^{(N)}(u)$ is the approximating Fourier polynomial $t^{(N)}(x)$. We see that the approximation of t(x) by $t^{(N)}(x)$ has the effect of quantizing the distribution T(u) into the discrete set of delta functions $T^{(N)}(u)$. We also note that the bandwidth of $t^{(N)}(x)$ is roughly the same as that of t(x), i.e., $T^{(N)}(u) = 0$ for $|u| > \beta$.

Equation (66) indicates that $t^{(N)}(x)$'s period L_o, and its degree N, are linearly related and in the limit, $L_o \longrightarrow \infty$ as $N \longrightarrow \infty$. But in such a situation

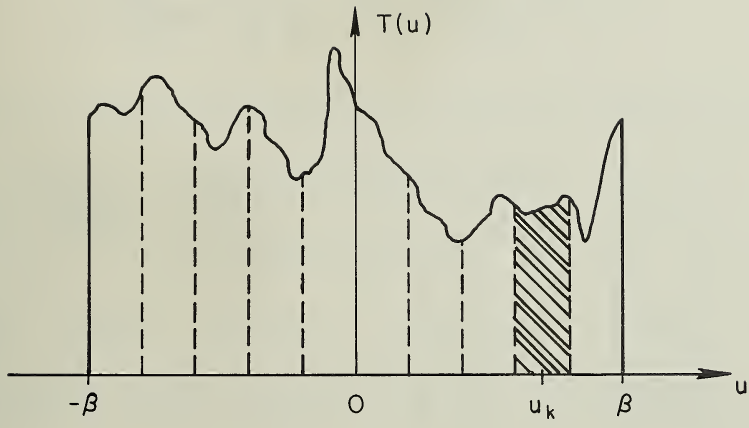

Figure 8a. Partitioning of Temperature Distribution Into N Sections

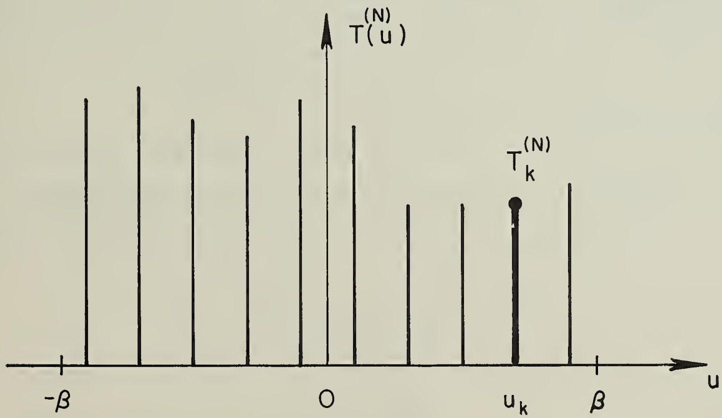

Figure 8b. Delta Function Representation of Temperature Distribution

$$\lim_{N \to \infty} \frac{\beta}{\pi N} \sum_{k=-N/2}^{N/2-1} T^{(N)}_k \; e^{ju_k x} = \frac{1}{2\pi} \int_{-\beta}^{\beta} T(u)\, e^{jux}\, du \tag{68}$$

where we have let

$$\begin{aligned} T_k^{(N)} &= T[(k+\tfrac{1}{2})\,\Delta u] \longrightarrow T(u) \\ u_k &= (k+\tfrac{1}{2})\,\Delta u \longrightarrow u \\ \text{and} \qquad \Delta u &= 2\beta/N \longrightarrow du \end{aligned} \tag{69}$$

Thus by <u>formally</u> going to the limit as N approaches infinity the approximating polynomial approaches the actual Fourier transform of the source distribution T(u).

Of course, in practice this is impossible since only a finite number of coefficients can be determined. However for a given aperture it is reasonable to believe that by making N (and hence L_o) suitably large, the polynomial that results will, to some extent, continue to approximate t(x) for $|x| > L$ and hence increase the effective aperture of the antenna.

4.2 The Least Square Formulation

In the usual manner we require, when we approximate t(x) by $t^{(N)}(x)$, that the average of the squared modulus of the error be a minimum:

$$\Delta = \int_{-L}^{L} | t(x) - t^{(N)}(x) |^2 \, dx = \min \tag{70}$$

or

$$\Delta = \int_{-L}^{L} \left| t(x) - \frac{\beta}{\pi N} \sum_{k=-N/2}^{N/2-1} T_k^{(N)} \, e^{ju_k x} \right|^2 dx = \min \tag{71}$$

Then we require that

$$\frac{\partial \Delta}{\partial T_k^{(N)}} = 0 \text{ for } k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \tag{72}$$

This yields N normal equations which in matrix form can be written as

$$\begin{bmatrix} S_{-\frac{N}{2},-\frac{N}{2}} & S_{-\frac{N}{2},-\frac{N}{2}+1} & S_{-\frac{N}{2},-\frac{N}{2}+2} & \cdots \\ S_{-\frac{N}{2}+1,-\frac{N}{2}} & S_{-\frac{N}{2}+1,-\frac{N}{2}+1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & S_{\frac{N}{2}-1,\frac{N}{2}-1} \end{bmatrix} \begin{bmatrix} T^{(N)}_{-\frac{N}{2}} \\ T^{(N)}_{-\frac{N}{2}+1} \\ \vdots \\ T^{(N)}_{\frac{N}{2}-1} \end{bmatrix} = \begin{bmatrix} T^{(N)}_{0\,-\frac{N}{2}} \\ T^{(N)}_{0\,-\frac{N}{2}+1} \\ \vdots \\ T^{(N)}_{0\,\frac{N}{2}-1} \end{bmatrix} \tag{73}$$

or

$$S\, T^{(N)} = T_o^{(N)} \tag{74}$$

where S is an $N \times N$ square matrix, and $T^{(N)}$ and $T_o^{(N)}$ are N dimensional column vectors; the matrix element in the kth row and ℓth column is

$$\begin{aligned} S_{k\ell} &= \frac{\sin (u_k - u_\ell) L}{(u_k - u_\ell) L} \\ &= \frac{\sin (k - \ell) \frac{2\beta}{N} L}{(k - \ell) \frac{2\beta}{N} L} \end{aligned} \tag{75}$$

The kth element of $T_o^{(N)}$ is given by

$$T_{0k}^{(N)} = \frac{\pi N}{2\beta} \int_{-L}^{L} t(x)\, e^{-ju_k x}\, dx$$

$$= \frac{\pi N}{2\beta} T_o(u_k) \tag{76}$$

where $T_o(u_k)$ is the value of the <u>principal solution</u>* sampled at $u_k = (k + 1/2)\, 2\,\beta/N$.

<u>4.3 Critical Value of $N = N_o$</u>

The formula for the $T_o^{(N)}$ elements is quite similar to that for the coefficients of an orthogonal Fourier series expansion of the complex function t(x) on the interval $-L \leq x \leq L$. Indeed if the number of terms N of our polynomial $t^{(N)}(x)$ is four times the length of the aperture in wavelengths ($N = N_o = 4L/\lambda$), then the matrix elements are given by

$$S_{k\ell} = \frac{\sin\,(k - \ell)\,\pi}{(k - \ell)\,\pi} = \delta_{k\ell} \tag{77}$$

where

$$\delta_{k\ell} = 1 \text{ if } k = \ell$$
$$= 0 \text{ if } k \neq \ell$$

is the Kronecker delta.

In this case the approximating polynomial is the usual complex Fourier series whose individual functions are orthogonal on the measurement interval $-L \leq x \leq L$. The matrix S has become an identity matrix I and from Equation (74) one can write

*See Equation (31), Chapter 2.2.

$$T_k^{(N_o)} = T_{0k}^{(N_o)} \quad \text{for } N_o = \frac{4L}{\lambda} \text{ and } k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \tag{78}$$

or

$$T_k^{(N_o)} = \frac{\pi N_o}{2\beta} \int_{-L}^{L} t(x)\, e^{-ju_k x}\, dx \tag{79}$$

which of course is

$$T_k^{(N_o)} = \frac{\pi N_o}{2\beta} T_o(u_k) \tag{80}$$

where $T_o(u_k)$ is the principal solution sampled at $u = u_k$.

It is easy to show for this critical case of $N = N_o = 4L/\lambda$ that the spacing between sample points u_k is that which is specified by the sampling theorem[17]. The elements of the solution vector $T^{(N_o)}$ are, except for a constant factor, the samples of the principal solution in the visible range. Using the sampling theorem one can write

$$T_o(u) \simeq \frac{2\beta}{\pi N_o} \sum_{k=-N_o/2}^{N_o/2-1} T_k^{(N_o)} \frac{\sin(Lu - k\pi)}{Lu - k\pi}, \qquad N_o = 4L/\lambda \tag{81}$$

or

$$T_o(u) \simeq \sum_{k=-N_o/2}^{N_o/2-1} T_o(u_k) \frac{\sin(Lu - k\pi)}{Lu - k\pi}, \qquad N_o = 4L/\lambda \tag{82}$$

Of course this does not give the principal solution exactly (since there are only N_o terms) but since $T_o(u) \simeq 0$ for $|u| > \beta$ it will usually be a very good approximation to $T_o(u)$, especially for large N_o.

4.4 Case of N Larger than the Critical Value

If N, the degree of the approximating polynomial, is now increased beyond N_o to say $2N_o$, which incidentally also doubles the period L_o of

the polynomial, then the least squares fit to the function $t(x)$ will clearly be improved on the measurement interval and, hopefully, we expect it to continue to fit the curve for values of x which lie just outside the interval.

Physically, we can think of the $2N_o$ coefficients, $T_k^{(2N_o)}$, as estimates of the average value of $T(u)$ on the corresponding intervals of u. Since there are twice as many distinct intervals as before the effective resolution has been doubled. However the solution is not identical to that of the principal solution of an antenna which is twice as large. In the latter case the $2N_o = 4(2L/\lambda)$ estimates are obtained by fitting the curve over its entire length of $4L$ whereas in the former the fitting is done over the original length of $2L$. The result is a better fit on this interval but a worse fit on the $[L, 2L]$ interval.

From the practical viewpoint the increase of N beyond N_o causes the coefficient matrix to revert to a general form and as a result the coefficients $T_k^{(N)}$ must be obtained by matrix inversion.

$$T^{(N)} = S^{-1} T_o^{(N)} \tag{83}$$

From Equation (76) we see that when N is increased the elements of $T_o^{(N)}$ are obtained by merely taking more closely spaced samples of the principal solution $T_o(u)$ in the visible range. But according to the sampling theorem, for $N > N_o$ these samples are no longer completely independent. This lack of independence is indeed the reason why the coefficient matrix S has reverted to its general form and in the next section it will be shown that as N is increased beyond N_o, S becomes more and more ill-conditioned as a result of this reduction of independence.

5. THE PROPERTIES OF THE COEFFICIENT MATRIX FOR $N > N_o$

In this chapter it will be shown when N and hence L_o are increased beyond their critical values that not only do the elements of the $T_o^{(N)}$ vector lose some of their independence but that the individual functions, $e^{ju_k x}$, of the approximating polynomial rapidly depart from their mutual orthogonality which occurs for $N = N_o$. A measure of the linear dependence of these functions is the Gram determinant[18] or Gramian which quickly reduces to a very small value when N exceeds N_o. In this particular case the Gramian is the determinant of S and it is well known that matrices with very small determinants are usually ill-conditioned[19]. As a result S possesses an inverse, S^{-1}, whose elements are large in magnitude and of alternating sign. This means that the solution vector $T^{(N)}$ involves the differences of large and usually nearly equal numbers and hence is very sensitive to errors.

5.1 The Gramian as a Measure of Independence of the Approximating Functions

In a manner which is similar to that used for measuring the independence of vectors one can calculate the Gramian or Gram determinant[18] associated with the approximating functions and the interval $-L \leq x \leq L$.

We define the Hermitian scalar product

$$\langle e_k(x)\, e_\ell^*(x)\rangle = \frac{1}{2L}\int_{-L}^{L} e^{ju_k x}\, e^{-ju_\ell x}\, dx$$

$$= \frac{\sin\,(u_k - u_\ell)\,L}{(u_k - u_\ell)\,L} \tag{84}$$

and interpret it as the cosine of the "angle" in Hilbert space between the functions $e_k(x) = e^{ju_k x}$ and $e_\ell(x) = e^{ju_\ell x}$. The Gramian Γ is the determinant of the following matrix of such scalar products.

$$\Gamma = \det \begin{bmatrix} \langle e_{-\frac{N}{2}}(x)\, e^*_{-\frac{N}{2}}(x) \rangle & \langle e_{-\frac{N}{2}}(x)\, e^*_{-\frac{N}{2}+1}(x) \rangle & \cdots \\ \langle e_{-\frac{N}{2}+1}(x)\, e^*_{-\frac{N}{2}}(x) \rangle & \langle e_{-\frac{N}{2}+1}(x)\, e^*_{-\frac{N}{2}+1}(x) \rangle & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \langle e_{\frac{N}{2}-1}(x)\, e^*_{\frac{N}{2}-1}(x) \rangle \end{bmatrix} \tag{85}$$

But in this case we see from Equations (75) and (84) that

$$\Gamma = \det S \tag{86}$$

It will be recalled that for $N = N_o$ the matrix S is an identity matrix and

$$\Gamma = \det I = 1 \tag{87}$$

Consequently the approximating functions are all mutually orthogonal since the cosine of the "angle" between any distinct two of them is zero.

Now as N increases beyond N_o to say $2N_o = 8L/\lambda$, the coefficient matrix becomes

$$S = \begin{bmatrix} 1 & \frac{\sin \pi/2}{\pi/2} & 0 & \frac{\sin 3\pi/2}{3\pi/2} & 0 & \cdots \\ \frac{\sin \pi/2}{\pi/2} & 1 & \frac{\sin \pi/2}{\pi/2} & 0 & \frac{\sin 3\pi/2}{3\pi/2} & \cdots \\ 0 & \frac{\sin \pi/2}{\pi/2} & 1 & \frac{\sin \pi/2}{\pi/2} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{88}$$

For $L = \lambda/2$ it will be truncated to $N = 4$ rows and columns; for $L = \lambda$ there will be 8 and so on. It is of interest to note that it is a striped matrix since not only are all the elements of the main diagonal equal but all elements of any off-diagonal are also equal. This is known as a Toeplitz form[20].

The gram determinants, as functions of the original measurement intervals in wavelengths L/λ, are plotted in Figure 9. Note that there is a very rapid falling off as L increases. From the practical viewpoint this means that it will be easier to extrapolate by 100 percent from a small aperture than from a larger one. In Figure 10 the value of the Gramian Γ is plotted as a function of the normalized effective aperture $\Lambda = N/N_o = L_o/L$ with the original measurement aperture L/λ as a parameter. It can be seen that the Gramian rapidly falls from 1 at $\Lambda = 1$ (where $N = N_o$) to very small values as Λ increases. Again the value of Γ for a given $\Lambda > 1$ is much smaller for large L than for small.

Since the smallness of Γ is a measure of the lack of independence of the functions $e_k(x)$ on the interval, we see that any significant extrapolation that is performed with this method must be done with functions which are strongly dependent.

5.2 The Inverse Matrix S^{-1}

Matrices whose determinants are very small usually possess inverses whose elements are extremely large. The coefficient matrix S is no exception and for $\Lambda \gg 1$, S is very ill-conditioned, i.e., the elements of S^{-1} are astronomical. For example if $L = \lambda$ and $N = 8$, $\Lambda = 2$ and the inverse of S is

$$S^{-1} = \begin{bmatrix} 93. & -329. & 648. & -867. & 834. & -574. & 265. & -66. \\ & 1{,}205. & -2{,}423. & 3{,}296. & -3{,}213. & 2{,}242. & -1{,}050. & 265. \\ & & 4{,}947. & -6{,}808. & 6{,}710. & -4{,}732. & 2{,}242. & -574. \\ & & & 9{,}468. & -9{,}423. & 6{,}710. & -3{,}213. & 834. \\ & & & & 9{,}468. & -6{,}808. & 3{,}296. & -867. \\ & & & & & 4{,}947. & -2{,}423. & 648. \\ & \multicolumn{2}{l}{S^{-1} \text{ is symmetric}} & & & & 1{,}205. & -329. \\ & \multicolumn{2}{l}{S^{-1}_{jk} = S^{-1}_{kj}} & & & & & 93. \end{bmatrix} \tag{89}$$

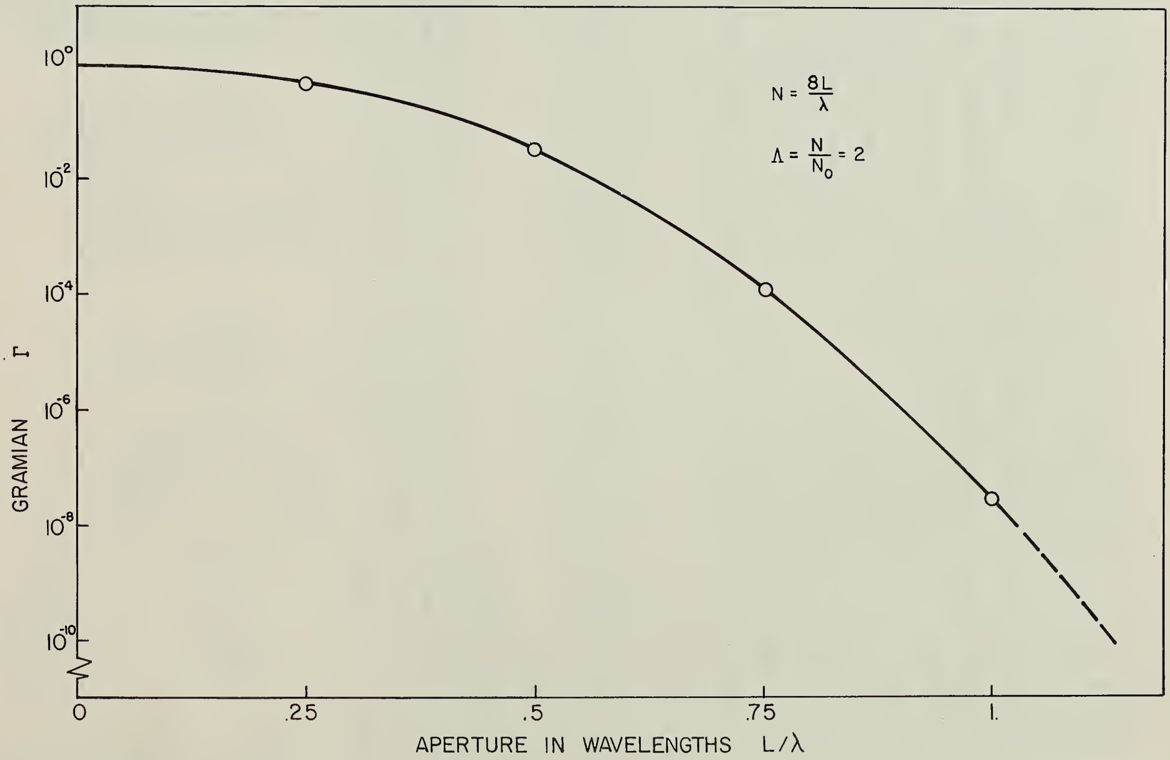

Figure 9. Value of Gramian Γ as a Function of Aperture Size in Wavelengths L/λ, for the Case of Normalized Effective Aperture $\Lambda = N/N_o = 2$.

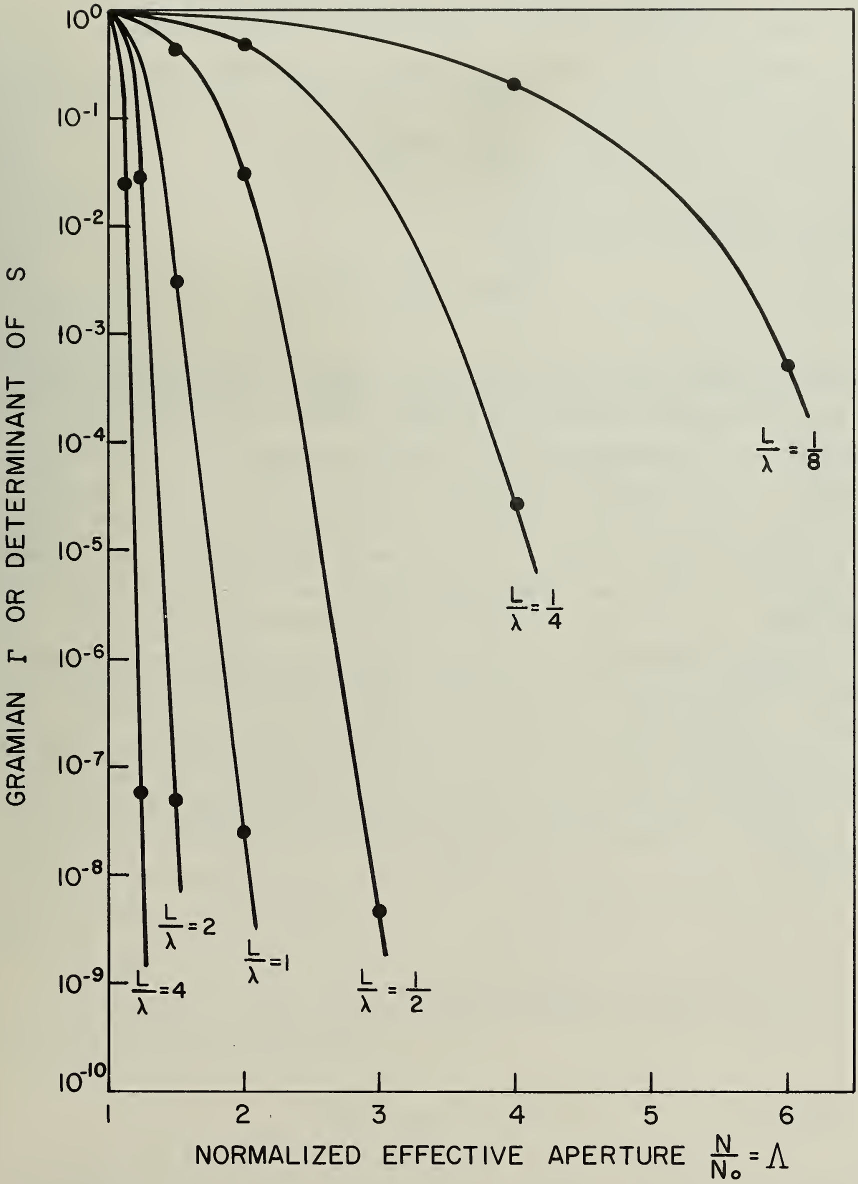

Figure 10. Value of Gramian Γ as a Function of the Normalized Effective Aperture Λ with Actual Aperture in Wavelengths as a Parameter

The smallest element (in magnitude) is 66.00... and the largest is 9,468.5... This is none too good, but if N is then increased to 12 and L/λ is held fixed at 1, then $\Lambda = 3$ and the smallest and largest (in magnitude) elements of the inverse matrix in this case are 4.89×10^5 and 3.39×10^{10} respectively.

However the size of the elements alone does not necessarily imply that the process is an ill-conditioned one. Of equal importance is the fact that the elements of S^{-1} alternate in sign as can be seen in Equation (89). Because of this the solution vector

$$T^{(N)} = S^{-1} T_o^{(N)} \tag{90}$$

involves the small differences of very large numbers. To preserve accuracy it will be necessary to specify the elements of both S^{-1} and of $T_o^{(N)}$ to a large number of decimal places. This can easily be done for S^{-1} since the elements of S are expressed by the exact formula of Equation (75). In practice one can specify them to say twelve decimal places and then invert S on a high speed digital computer. The process of inversion will involve round off errors but in most cases the elements of S^{-1} can be calculated to at least eight or nine decimal digits of accuracy. However the vector $T_o^{(N)}$ is obtained from measurements of t(x) which can in most cases be performed with a maximum precision and accuracy of only about three or four decimal places.

The weak point of the process is therefore the sensitivity of the solution vector $T^{(N)}$ to small errors in the measurement vector $T_o^{(N)}$. In the next section a statistical analysis of the effect of measurement error is presented.

6. ERROR ANALYSIS FOR ILL-CONDITIONED MATRICES

Let the data vector $\overline{T}_o^{(N)}$ in the matrix equation

$$S\ \overline{T}^{(N)} = \overline{T}_o^{(N)} \tag{91}$$

be written as

$$\overline{T}_o^{(N)} = T_o^{(N)} + \delta\ T_o^{(N)} \tag{92}$$

where $T_o^{(N)}$ is the exact value of $\overline{T}_o^{(N)}$ and $\delta\ T_o^{(N)}$ is the unavoidable error due to measurement. Then we can write

$$S \left\{ T^{(N)} + \delta\ T^{(N)} \right\} = T_o^{(N)} + \delta\ T_o^{(N)} \tag{93}$$

where

$$\delta\ T^{(N)} = S^{-1}\ \delta\ T_o^{(N)} \tag{94}$$

is the error in the calculated vector whose true value is $T^{(N)}$. Here we have assumed that we can obtain an inverse matrix S^{-1} which is exact. In practice, of course, the process of inverting S will involve round-off errors. However, the elements of S can be specified to an arbitrarily large number of decimal places and if the inversion is done by a large digital computer* with a capacity of 10 or 12 decimal digits, then the error in S^{-1} is negligible compared to the error in $\overline{T}_o^{(N)}$ which is due to measurement and which probably is no less than one part in ten thousand or 4 decimal digits accuracy.

The $\overline{T}_o^{(N)}$ vector is not itself a measured quantity being obtained from the measured function $\overline{t}(x)$ by means of Equation (76) of Chapter 4.2. We can write the measured function as

$$\overline{t}(x) = t(x) + \delta\ t(x) \tag{95}$$

*The University of Illinois Digital Computer, ILLIAC, was used in all numerical work.

where $\delta\, t(x)$ is the unavoidable error in the measurement of the true spectrum $t(x)$. For simplicity, let us assume that the source distribution $T(u)$ is uniform with $T(u) = T_A$, a positive real constant. In this case the measured function is

$$\bar{t}(x) = 2\beta\, T_A \frac{\sin \beta x}{\beta x} + \delta\, t(x) \tag{96}$$

where $2\beta\, T_A \frac{\sin \beta x}{\beta x}$ is the spatial frequency spectrum of $T(u) = T_A$.

The elements of the true $T_o^{(N)}$ vector can be calculated by means of the above mentioned equation. They are given (for N even) by

$$T_{0k}^{(N)} = T_A \left[\mathrm{Si} \left\{ \left[1 + (k + \tfrac{1}{2}) \tfrac{2}{N}\right] \beta L \right\} + \mathrm{Si} \left\{ \left[1 - (k + \tfrac{1}{2}) \tfrac{2}{N}\right] \beta L \right\} \right] \tag{97}$$

where

$$\mathrm{Si}(y) = \int_0^y \frac{\sin x}{x}\, dx$$

is the sine integral.

Now it is reasonable to assume that the error function, $\delta\, t(x)$, in the measurement of $t(x)$ has statistically independent real and imaginary parts whose means are zero. Thus if $t(x) = t_1(x) + jt_2(x)$,

$$E \left\{ \delta\, t_1(x)\, \delta\, t_2(x) \right\} = 0 \tag{98}$$

$$E \left\{ \delta\, t_1(x) \right\} = E \left\{ \delta\, t_2(x) \right\} = 0 \tag{99}$$

We let the autocorrelation function for both $\delta\, t_1(x)$ and $\delta\, t_2(x)$ be given by

$$R_\delta(x) = E \left\{ \delta\, t_1(x_1)\, \delta\, t_1(x_2) \right\} = E \left\{ \delta t_2(x_1)\, \delta t_2(x_2) \right\} \tag{100}$$

If one makes the usual assumption that the error power spectrum is flat (white noise) with a power per unit bandwidth of $N_o^2/2$ and a bandwidth of M radians per meter, then the autocorrelation function of $\delta t_1(x)$ and $\delta t_2(x)$ is

$$R_\delta(x) = \frac{M N_o^2}{2\pi} \frac{\sin Mx}{Mx} = \frac{\sigma_t^2}{2} \frac{\sin Mx}{Mx} \tag{101}$$

where $x = x_1 - x_2$. In what follows it will be assumed that the error spectrum is much broader than the signal spectrum, $M \gg \beta$.

In the appendix it is shown that the expected value of each element of the error vector $\delta T_o^{(N)}$ is zero,

$$E\left\{\delta T_{0k}^{(N)}\right\} = 0 \qquad \text{for} \qquad k = \frac{-N}{2}, \ldots \frac{N}{2} - 1 \tag{102}$$

and the covariance of the elements $\delta T_{0\ell}^{(N)}$ and $\delta T_{0j}^{(N)}$ is

$$E\left\{\delta T_{0\ell}^{(N)}\ \delta T_{0j}^{(N)}\right\} = \frac{\pi L\, \sigma_t^2}{M} \frac{\sin (\ell - j) \frac{4\pi}{N} \frac{L}{\lambda}}{(\ell - j) \frac{4\pi}{N} \frac{L}{\lambda}} \tag{103}$$

We see that $\pi L\, \sigma_t^2/M$ is the variance of the individual elements of the error vector.

With this statistical information we can now derive expressions for the mean and variance of the error in the solution vector $\overline{T}^{(N)}$.

$$E\left\{\delta T^{(N)}\right\} = E\left\{S^{-1} \delta T_o^{(N)}\right\} = 0 \tag{104}$$

The average error is zero. To calculate the variance we write

$$E\left\{\delta T^{(N)^T} \delta T^{(N)}\right\} = E\left\{\left[S^{-1} \delta T_o^{(N)}\right]^T S^{-1} \delta T_o^{(N)}\right\} \tag{105}$$

where the superscript ^{T} denotes the transpose of the vector. A typical element of the resulting sum is

$$E\left\{\delta T_k^{(N)^2}\right\} = E\left\{\sum_{\ell=-N/2}^{N/2-1} S_{k\ell}^{-1}\,\delta T_{o\ell}^{(N)} \sum_{j=-N/2}^{N/2-1} S_{kj}^{-1}\,\delta T_{oj}^{(N)}\right\}$$

$$= \sum_{\ell=-N/2}^{N/2-1} \sum_{j=-N/2}^{N/2-1} S_{k\ell}^{-1} S_{kj}^{-1}\, E\left\{\delta T_{o\ell}^{(N)}\,\delta T_{oj}^{(N)}\right\} \tag{106}$$

where S_{kj}^{-1} is the kjth element of the inverse matrix S^{-1}. From Equation (103) we have

$$E\left\{\delta T_k^{(N)^2}\right\} = \sum_{\ell=-N/2}^{N/2-1} \sum_{j=-N/2}^{N/2-1} S_{k\ell}^{-1} S_{kj}^{-1} \frac{\pi L \sigma_t^2}{M} \frac{\sin(\ell-j)\frac{4\pi}{N}\frac{L}{\lambda}}{(\ell-j)\frac{4\pi}{N}\frac{L}{\lambda}} \tag{107}$$

But since $E\left\{\delta T_k^{(N)}\right\} = 0$ we can write down the variance of $\delta T_k^{(N)}$ as

$$\sigma^2_{\delta T_k^{(N)}} = \frac{\pi L}{M} \sum_{\ell=-N/2}^{N/2-1} \sum_{j=-N/2}^{N/2-1} S_{k\ell}^{-1} S_{kj}^{-1} \frac{\sin(\ell-j)\frac{4\pi}{N}\frac{L}{\lambda}}{(\ell-j)\frac{4\pi}{N}\frac{L}{\lambda}}\,\sigma_t^{\,2} \tag{108}$$

or

$$\sigma^2_{\delta T_k^{(N)}} = A_{kN}\,\sigma_t^2 \tag{109}$$

This equation shows by how much the error in measurement is amplified to give the error in the calculated data. There is an A_{kN} for each element of the $\delta T^{(N)}$ vector although only half are distinct due to the symmetry of the matrix. The total variance of the calculated error vector is the sum of the individual variances of its elements.

$$\sigma^2_{\delta T}(N) = \sum_{k=-N/2}^{N/2-1} \sigma^2_{\delta T_k}(N) = \sum_{k=-N/2}^{N/2-1} A_{kN}\,\sigma^2_t \tag{110}$$

or

$$\sigma^2_{\delta T}(N) = A_N\,\sigma^2_t \tag{111}$$

Geometrically we can think of the positive square root of the variance as the radius of a sphere* of uncertainty centered at the tip of the true solution vector $T^{(N)}$.

Parenthetically, we note from Equation (108) that the coefficients A_{kN} depend inversely on the bandwidth M of the error power spectrum. This apparent anomaly is due to our defining the variance of the measurement error as σ^2_t which can also be written as

$$\sigma^2_t = \frac{M\,N_o^2}{\pi} \tag{112}$$

We see that the M in this expression cancels the M in the A_{kN} coefficient and Equation (108) becomes

$$\sigma^2_{\delta T_k}(N) = \sum_{\ell=-N/2}^{N/2-1} \sum_{j=-N/2}^{N/2-1} S^{-1}_{k\ell}\, S^{-1}_{kj}\, \frac{\sin[(\ell-j)\frac{4\pi}{N}\frac{L}{\lambda}]}{(\ell-j)\frac{4\pi}{N}\frac{L}{\lambda}}\, L\, N_o^2 \tag{113}$$

in which it can be seen that the calculated variance is proportional to the power per unit bandwidth N_o^2 of the measurement error function. However, Equation (108) is useful in showing how the calculation process acts as a filter which suppresses the high frequency components of the error spectrum. For example, if two measurements of t(x) are made, one of which oscillates rapidly about the correct value with variance σ^2_t and the other oscillates much more slowly about the correct value with the same variance, then the error in the calculated data will be much less for the first measurement

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*Since the A_{kN} in Equation (109) is not identical the actual shape of the uncertainty volume will be ellipsoidal.

than for the second. For a given measurement variance the calculated variance is inversely proportional to the bandwidth of the measurement error power spectrum.

Now it is important to know by how much the effective aperture can be increased (by increasing N) before the radius of the sphere of uncertainty exceeds, for example, one percent of the length of the true solution vector. Clearly this depends on the relative magnitudes of the signal and error functions, $t(x)$ and $\delta t(x)$, as well as the width of $\delta t(x)$'s power spectrum. In Figure 11 is shown a graph of $t(x)$ for the case of a uniform temperature distribution with a shaded area extending σ_t units above and below the curve. This indicates the size of the standard deviation which as a fraction of the value of $t(x)$ at $x = 0$ is

$$\sigma'_t = \frac{\sigma_t}{2\beta T_A} \tag{114}$$

For larger values of $|x|$ it is always greater than this. From Equation (97) we can calculate the value of the elements of the true data vector $T_o^{(N)}$ and then

$$T^{(N)} = S^{-1}\, T_o^{(N)} \tag{115}$$

is the value of the correct $T^{(N)}$ vector. Then we let

$$\sigma'_{\delta T^{(N)}} = \frac{\sigma_{\delta T^{(N)}}}{|T^{(N)}|} \tag{116}$$

be the normalized standard deviation of the calculated vector $\overline{T}^{(N)}$ about its true length, i.e., $\sigma'_{\delta T^{(N)}}$ is the radius of the sphere of uncertainty as a fraction of the length of the solution vector $T^{(N)}$. As mentioned previously the calculated variance depends on the bandwidth of the measured error power spectrum M which we assumed was much larger than β. In all numerical calculations it was assumed that

$$M = 10\,\beta \tag{117}$$

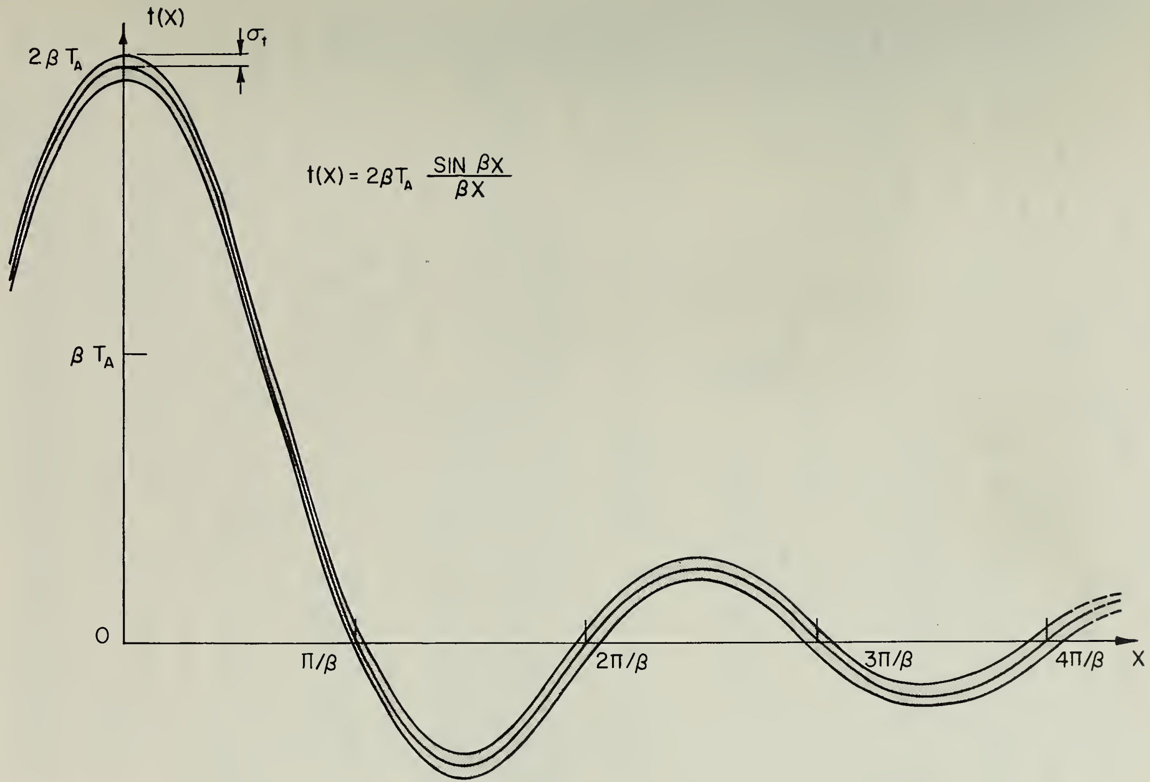

Figure 11. Spatial Frequency Spectrum of a Uniform Temperature Distribution with the Standard Deviation of the Measurement Error Indicated by the Width of the Shaded Area About the Curve

Admittedly this is rather arbitrary and in an actual application the value of M might be considerably different. However, if some estimate of the actual value of M is made, then one can use Equation (108) to calculate corrected value of the variance $\sigma^2_{\delta T^{(N)}}$. For example, if the estimate of M was 20 β, then since $\sigma^2_{\delta T^{(N)}}$ is inversely proportional to M (Equation 108) the actual variance is one half of that which is calculated for M = 10 β. In general, to obtain the actual variance corresponding to a value of M = Wβ one can use the following simple formula

$$\sigma^2_{\delta T^{(N)}}\Big|_{M=W\beta} = \frac{10}{W}\, \sigma^2_{\delta T^{(N)}}\Big|_{M=10\beta} \tag{118}$$

and

$$\sigma^2_{\delta T^{(N)}}\Big|_{M=10\beta}$$

is the variance obtained in all of the following calculations.

As a typical example the case of L = λ and N = 8 will be considered in detail. The inverse of the coefficient matrix is given by Equation (89). The variance of the kth element of $\delta T^{(8)}$ is

$$\sigma^2_{\delta T_k^{(8)}} = \frac{\pi L}{10\beta} \sum_{\ell=-4}^{3} \sum_{j=-4}^{3} S^{-1}_{k\ell}\, S^{-1}_{kj}\, \frac{\sin\, (\ell-j)\frac{\pi}{2}}{(\ell-j)\frac{\pi}{2}}\, \sigma^2_t \tag{119}$$

But from Equation (114) we substitute for σ^2_t to obtain

$$\sigma^2_{\delta T_k^{(8)}} = \frac{2\pi}{5}\, \beta L \sum_{\ell=-4}^{3} \sum_{j=-4}^{3} S^{-1}_{k\ell}\, S^{-1}_{kj}\, \frac{\sin\, (\ell-j)\frac{\pi}{2}}{(\ell-j)\frac{\pi}{2}}\, \sigma_t^{\,2}\, T_A^2 \tag{120}$$

This can be evaluated for the 8 values of k and for k = 1 one obtains

$$\sigma^2_{\delta T_1^{(8)}} = \frac{4\pi^2}{5} (4{,}947.14)\ \sigma_t'^2\ T_A^{\ 2}$$
$$= \quad 39{,}061.0\ \sigma_t'^2\ T_A^{\ 2} \tag{121}$$

By means of Equation (97) and (115) the value of $T_1^{(8)}$ can be obtained and the normalized standard deviation of the solution error about its true value is

$$\sigma'_{\delta T_1^{(8)}} = \frac{\sigma_{\delta T_1^{(8)}}}{T_1^{(8)}} = \frac{197.8\ \sigma_t' T_A}{1.9862\ T_A} = 99.8\ \sigma_t' \tag{122}$$

This normalized error line is plotted in Figure 12 along with those of the other 7 elements of the vector. Note that only half are distinct. The graph shows that if the normalized standard deviation of the measurement error σ_t' is .01 the normalized standard deviation of the calculated error varies from .156 to as much as 1.955. Conversely if one wants, on the average, a certain accuracy in all of the calculated results the measurement accuracy must be about 200 times greater than the calculated accuracy. An error line for the vector $\overline{T}^{(8)}$ as a whole is also shown in Figure 12. It is the heavy line with a slope of 110.5. It indicates that if one requires that the normalized radius of the sphere of uncertainty of the calculated $\overline{T}^{(8)}$ vector be less than one percent then the measured data must have a normalized standard deviation of (.01/110.5) 100 = .00903 percent. Thus we see that even for the seemingly modest increase in effective aperture of from one to two wavelengths ($\Lambda = 2$) the method of processing the data is such that a one hundred fold increase in relative error is to be expected.

This ratio of calculated error to measurement error is plotted in Figure 13 as a function of the effective aperture Λ with L/λ, the original aperture size in wavelengths, as a parameter. It can be seen that not only does $\sigma'_{\delta T^{(N)}} \big/ \delta_t'$ increase with increasing Λ but it increases as the original

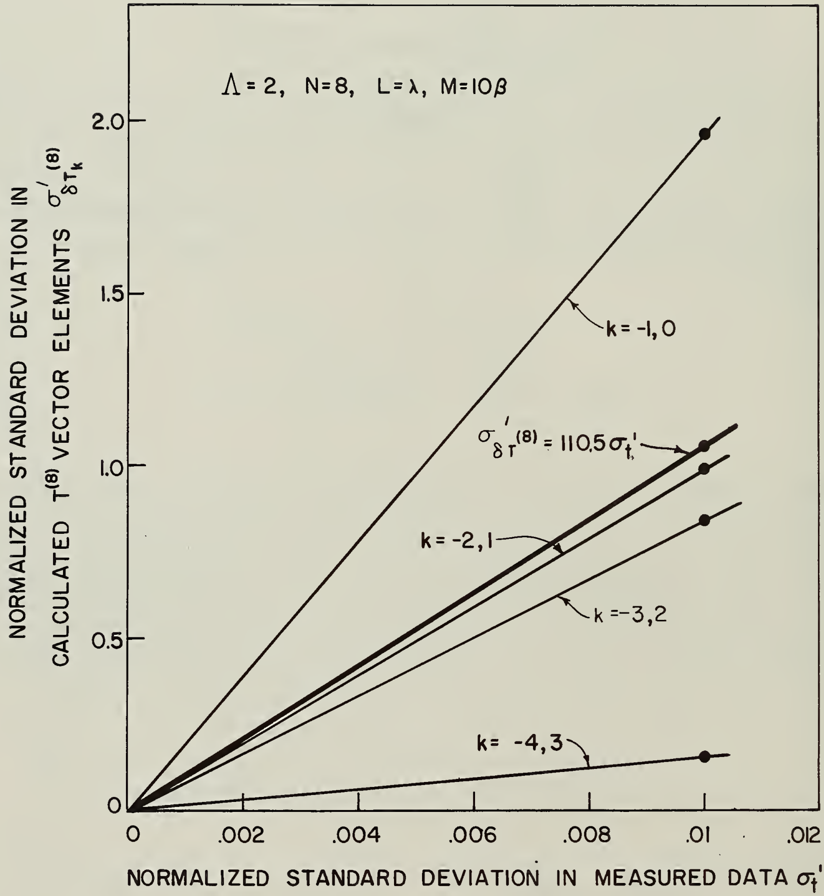

Figure 12. Normalized Standard Deviation in the Elements of the Calculated $\overline{T}^{(8)}$ Vector as a Function of the Normalized Standard Deviation in the Measured Data for $L = \lambda$, $N = 8$, $\Lambda = 2$.

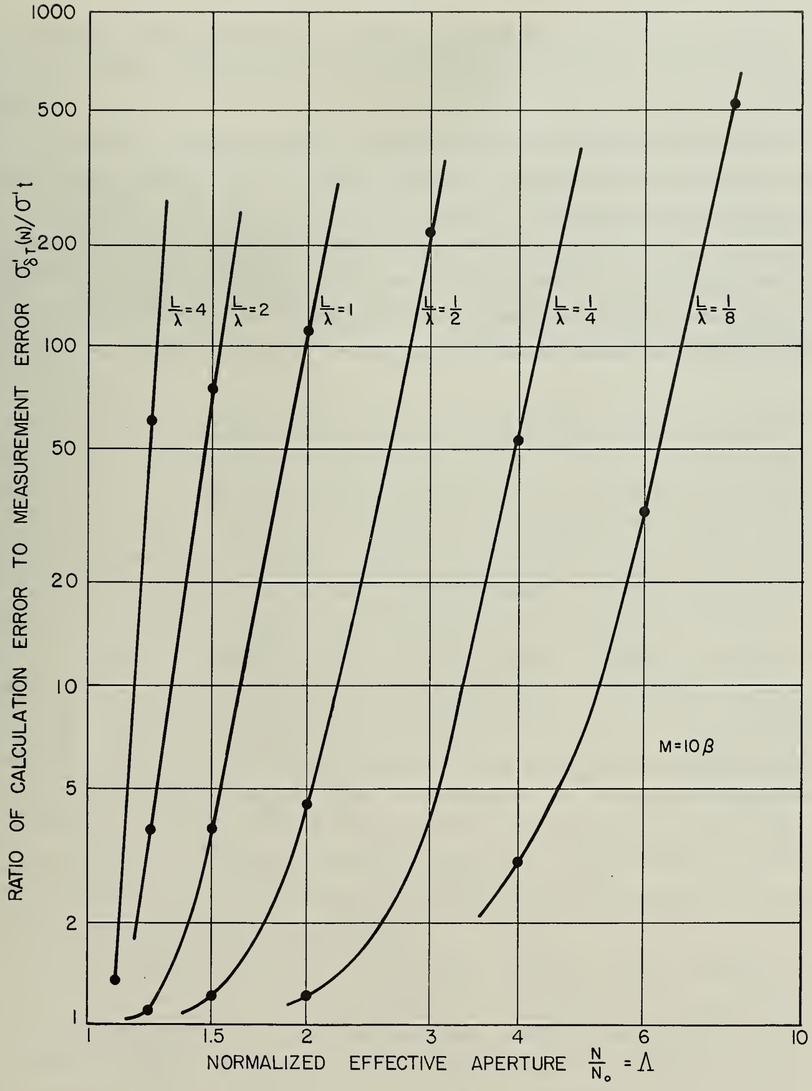

Figure 13. Ratio of Calculation Error to Measurement Error as a Function of the Normalized Effective Aperture with the Actual Aperture in Wavelengths as a Parameter

aperture size increases. For example, if Λ is fixed at 2 the value of $\sigma'_{\delta T}(N) / \sigma'_t$ increases from 1.225 for $L/\lambda = 1/4$, to 4.41 for $L/\lambda = 1/2$, and to 110.5 for $L/\lambda = 1$. On the other hand, if one can accept a value of $\sigma'_{\delta T}(N) / \sigma'_t$ of 100, then the normalized effective aperture will range from 6.6 for $L/\lambda = 1/8$ to about 1.3 for $L/\lambda = 4$.

It is quite evident that such a process is of negligible practical value since for moderately large antennas ($L/\lambda > 10$) the amount of useful increase in effective aperture would be very small. For very small antennas ($L/\lambda < 1/8$) the process does cause a significant increase in effective aperture and it is only in this area, i.e., very low frequency systems where the apertures are necessarily small in terms of wavelengths, that there might be a use for the process.

7. AN EXAMPLE OF THE INCREASED RESOLUTION OF AN ANTENNA AS A RESULT OF THE DATA PROCESSING

Although it has been shown that the proposed process is too sensitive to errors in the measured data to be of much practical value, we will (for academic purposes) consider briefly the increased resolution that is obtained in the case of $L = \lambda$ and $\Lambda = N/N_o = 2$. As shown in Chapter 6 the normalized average solution error for this case is roughly one hundred times larger than the normalized average error in the measured data. Consequently, for the results that follow it has been assumed that the measurement accuracy was about one part in ten thousand. This leaves an average error in the calculated data of about one percent. This corresponds to the usual accuracy of the graphical representation of radiation patterns in which form the results will be given.

Thus in Figure 14 the dotted curve is the principal solution $T_o(u)$ for a unit point source on the $u = 0$ direction. The solid curve is that which results when the elements of the solution vector $T_k^{(8)}$ are taken as the estimates of the average value of $T(u)$ at intervals $u_k = 2\beta/8$ and a smooth curve is passed through these 8 points. We note that the main lobe of the latter pattern is about one half the width of the principal solution's main lobe. In Figure 15 are plotted the corresponding curves for the case of a point source located at $u = \beta/\sqrt{2}$. We note that the curve $T^{(8)}(u)$ has a main lobe which is only about one third as wide as that of $T_o(u)$ but has much larger side lobes. Comparing the two cases of point sources at $u = 0$ and $u = \beta/\sqrt{2}$ we see that $T_o(u)$ is translation invariant, i.e., except for a translation to the right of $\beta/\sqrt{2}$ units its shape is unchanged. $T^{(8)}(u)$ is not translation invariant, however; $T^{(8)}(u)$ for the point source at $u = 0$ is not just a shifted version of $T^{(8)}(u)$ for the same source at $u = \beta/\sqrt{2}$. Had the error sensitivity not vitiated the process to begin with this would have been another difficulty in any practical application.

Finaly, we consider the case of two point sources of equal strength located at $u = -\beta/4$ and $u = \beta/4$ respectively. In Figure 16 is shown the principal solution $T_o(u)$ for the one wavelength aperture together with the $\Lambda = 2$ solution $T^{(8)}(u)$. Although the two sources are far from being resolved by the single lobed principal solution, the $\Lambda = 2$ solution has two distinct peaks which show quite closely the locations oı the two sources whose true locations are represented by delta functions.

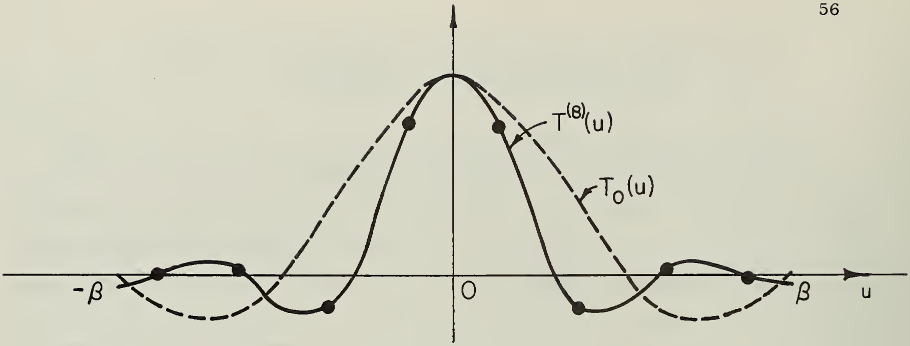

Figure 14. Principal Solution Pattern $T_o(u)$ and $\Lambda = 2$ Pattern $T^{(8)}(u)$ for a One Wavelength Aperture and a Point Source at u = 0

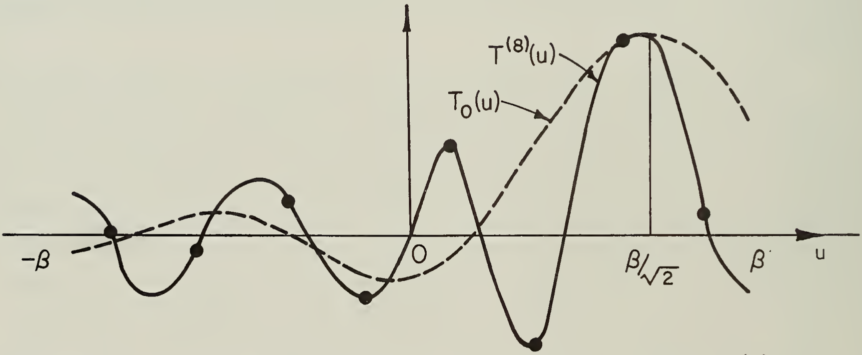

Figure 15. Principal Solution Pattern $T_o(u)$ and $\Lambda = 2$ Pattern $T^{(8)}(u)$ for a One Wavelength Aperture and a Point Source at $u = \beta/\sqrt{2}$

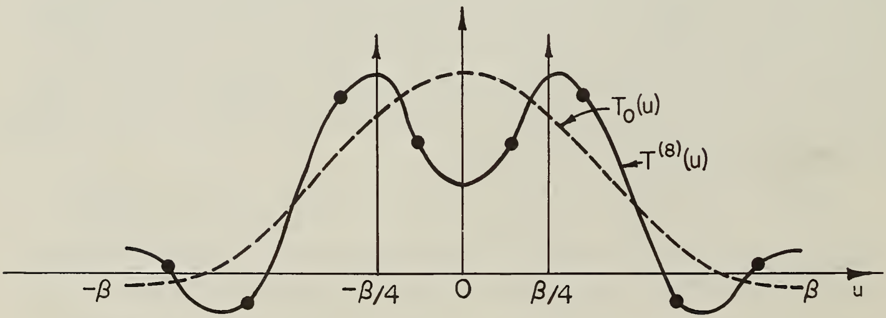

Figure 16. Principal Solution Pattern $T_o(u)$ and $\Lambda = 2$ Pattern $T^{(8)}(u)$ for a One Wavelength Aperture and Two Equal Point Sources at $u = -\beta/4$ and $u = \beta/4$ Respectively.

8. CONCLUSIONS

In this report it has been established that although a finite antenna acts as a perfect low pass filter of the spatial frequency spectrum of remote sources, it is theoretically possible to deduce the entire frequency spectrum from the output of the antenna. To do this we noted that since the spectrum is a band-limited analytic function, its values for the frequencies outside the pass band could be obtained by a process of analytic continuation of the function from within the band where it could be measured.

However, in the presence of measurement error it has been shown that very little meaningful extrapolation is possible, at least with the method described in this report. The general lack of success of this and several other proposed methods[7,8,9] would indicate that although the analytic function $t(x)$ on the interval $|x| \leq L$ contains an infinite amount of information (if it could be measured exactly), in practice it yields a negligible amount of information about its values outside the interval and it can be adequately described within the interval by $N_o = 4L/\lambda$ numbers. This, incidentally, is the same number that is required by the Shannon Sampling Theorem but the result is obtained here by a different approach.

It has also been shown that the process of expanding a function on a finite interval by an orthogonal Fourier series is not only very convenient (since $S = I$) but it is the only one of any practical value since the expansion of the non-orthogonal series is accompanied by the extreme error sensitivity described in Chapters 5 and 6 of this report.

Finally, we note the similarity between this problem and that of a supergain antenna and assert that although the data processing approach does not necessitate having antennas with relatively large reactive fields, it does have in common with supergain antennas an extreme sensitivity to errors in the physical parameters of the system. Since these errors cannot be made arbitrarily small, the data processing system is inherently as unstable as the conventional supergain system.

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APPENDIX

STATISTICAL PROPERTIES OF THE ERROR VECTOR $\delta T_o^{(N)}$

The kth element of the data error vector is given by*

$$\delta T_{ok}^{(N)} = \int_o^L \left\{ \delta t_1(x_1) \cos\left[(k+\tfrac{1}{2})\tfrac{2\beta}{N} x_1\right] + \delta t_2(x_1) \sin\left[(k+\tfrac{1}{2})\tfrac{2\beta}{N} x_1\right] \right\} dx_1 \quad \text{(A-1)}$$

where

$$\delta t(x) = \delta t_1(x) + j\, \delta t_2(x) \quad \text{(A-2)}$$

is a function representing the difference between the measured and the true value of $t(x)$. We assume that

$$E\left\{\delta t_1(x)\right\} = E\left\{\delta t_2(x)\right\} = E\left\{\delta t_1(x_1)\, \delta t_2(x_2)\right\} = 0 \quad \text{(A-3)}$$

The error has zero average and its real and imaginary parts are statistically independent. The autocorrelation functions of $\delta t_1(x)$ and $\delta t_2(x)$ are assumed to be the same and are given by

$$E\left\{\delta t_1(x_1)\, \delta t_1(x_2)\right\} = E\left\{\delta t_2(x_1)\, \delta t_2(x_2)\right\} = \frac{\sigma_t^2}{2} \frac{\sin Mx}{Mx} \quad \text{(A-4)}$$

or

$$R_\delta(x) = \frac{\sigma_t^2}{2} \frac{\sin Mx}{Mx} \quad \text{(A-5)}$$

where $x = x_1 - x_2$ and $M \gg \beta$. The error spectrum is flat and much broader than the spectrum of $t(x)$. Since the autocorrelation function is a function of x, the difference of the two sample points x_1 and x_2, the measurement error possesses the stationary property.

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*For convenience we have suppressed the constant factor $\pi N/\beta$ which would make Equation (A1) similar to Equation (76) (page 34).

AVERAGE VALUE OF $\delta T_{ok}^{(N)}$

The average or expected value of the error vector's kth element is

$$E\left\{\delta T_{ok}^{(N)}\right\} = \int_o^L \left[E\left\{\delta t_1(x_1)\right\} \cos\left[(k+\frac{1}{2})\frac{2\beta}{N}x_1\right]\right.$$

$$\left. + E\left\{\delta t_2(x_1)\right\} \sin\left[(k+\frac{1}{2})\frac{2\beta}{N}x_1\right]\right] dx_1 \tag{A-6}$$

But since the average values of $\delta t_1(x)$ and $\delta t_2(x)$ are zero (Equation A-3) we have

$$E\left\{\delta T_{ok}^{(N)}\right\} = 0 \qquad k = -\frac{N}{2}, \ . \ . \ . \ , \frac{N}{2} - 1 \tag{A-7}$$

The average value of the error vector $\delta T_o^{(N)}$ is zero.

COVARIANCE OF THE ERROR VECTOR

We now will determine the covariance of the elements δT_{ok}^{N} and $\delta T_{o\ell}^{(N)}$ of the error in the data vector

$$E\left\{\delta T_{ok}^{(N)} \ \delta T_{o\ell}^{(N)}\right\} = E\left\{\int_o^L [\delta t_1(x_1) \cos k_1x_1 + \delta t_2(x_1) \sin k_1x_1]\, dx_1 \ \cdot\right.$$

$$\left. \cdot \int_o^L [\delta t_1(x_2) \cos \ell_1x_2 + \delta t_2(x_2) \sin \ell_1x_2]\, dx_2\right\} \tag{A-8}$$

where $k_1 = (k + \frac{1}{2})\frac{2\beta}{N}$ and $\ell_1 = (\ell + \frac{1}{2})\frac{2\beta}{N}$.

The above can be written as

$$E\left\{\delta T_{ok}^{(N)}\ \delta T_{o\ell}^{(N)}\right\} = \int_o^L\int_o^L E\left\{\delta t_1(x_1)\ \delta t_1(x_2)\right\} \cos k_1x_1 \cos \ell_1x_2$$

$$+ E\left\{\delta t_1(x_1)\ \delta t_2(x_2)\right\} \cos k_1x_1 \sin \ell_1x_2$$

$$+ E\left\{\delta t_1(x_1)\ \delta t_1(x_2)\right\} \sin k_1x_1 \cos \ell_1x_2 \tag{A-9}$$

$$+ E\left\{\delta t_2(x_1)\ \delta t_2(x_2)\right\} \sin k_1x_1 \sin \ell_1x_2 \quad dx_1\ dx_2$$

Due to the statistical independence of $\delta t_1(x)$ and $\delta t_2(x)$ the two middle terms of the integrand are zero and by Equations (A-4) and (A-5) we have

$$E\left\{\delta T_{ok}^{(N)}\ \delta T_{o\ell}^{(N)}\right\} = \int_o^L\int_o^L R_\delta(x) \cos k_1x_1 \cos \ell_1x_2\ dx_1\ dx_2$$

$$+ \int_o^L\int_o^L R_\delta(x) \sin k_1x_1 \sin \ell_1x_2\ dx_1\ dx_2 \tag{A-10}$$

After much algebraic manipulation and by noting that $R_\delta(x) = R_\delta(-x)$, the above pair of double integrals can be reduced to

$$E\left\{\delta T_{ok}^{(N)}\ \delta T_{o\ell}^{(N)}\right\} = L\,\frac{\cos\left[(k_1-\ell_1)L\right] + 1}{(k_1-\ell_1)L}\int_o^L R_\delta(x)\left[\sin \ell_1x - \sin k_1x\right] dx$$

$$+ L\,\frac{\sin (k_1-\ell_1)L}{(k_1-\ell_1)L}\int_o^L R_\delta(x)\left[\cos \ell_1x + \cos k_1x\right] dx \tag{A-11}$$

Then by substituting for $R_\delta(x)$ from Equation (A-5) we get

$$E\left\{\delta T_{ok}^{(N)}\ \delta T_{ok}^{(N)}\right\} = L\,\frac{\cos\,[k_1-\ell_1)L]+1}{(k_1-\ell_1)L}\int_o^L \frac{\sigma_t^{\,2}}{2}\,\frac{\sin Mx}{Mx}\,[\sin \ell_1 x - \sin k_1 x]\,dx$$

$$+ L\,\frac{\sin\,[(k_1-\ell_1)L]}{(k_1-\ell_1)L}\int_o^L \frac{\sigma_t^{\,2}}{2}\,\frac{\sin Mx}{Mx}\,[\cos \ell_1 x + \cos k_1 x]\,dx \qquad \text{(A-12)}$$

$$= L\,\frac{\cos\,[(k_1-\ell_1)L]+1}{2(k_1-\ell_1)L}\,\frac{\sigma_t^{\,2}}{2M}\,\Big[C_i\left\{(M-\ell_1)L\right\} - C_i\left\{(M+\ell_1)L\right\}$$

$$- C_i\left\{(M-k_1)L\right\} + C_i\left\{(M+k_1)L\right\}\Big]$$

$$+ L\,\frac{\sin\,[(k_1-\ell_1)L]}{2(k_1-\ell_1)L}\,\frac{\sigma_t^{\,2}}{2M}\,\Big[S_i\left\{(M+\ell_1)L\right\} + S_i\left\{(M-\ell_1)L\right\}$$

$$+ S_i\left\{(M+k_1)L\right\} + S_i\left\{(M-k_1)\,L\right\}\Big] \qquad \text{(A-13)}$$

where

$$C_i(y) = \int_\infty^y \frac{\cos y_1}{y_1}\,dy_1$$

is the cosine integral. Now if $M \gg \beta$, then $M \gg \ell_1 = (\ell+1/2)\,2\beta/N$ and $M \gg k_1 = (k+1/2)\,2\beta/N$. The above expression then simplifies to

$$E\left\{\delta T_{ok}^{(N)}\ \delta T_{o\ell}^{(N)}\right\} \sim L\,\frac{\sin\,[(K-\ell)\frac{4\pi}{N}\frac{L}{\lambda}]}{(k-\ell)\frac{4\pi}{N}\frac{L}{\lambda}}\,\frac{\sigma_t^{\,2}}{2M}\,4\,\frac{\pi}{2}$$

$$\simeq \frac{\pi L\,\sigma_t^{\,2}}{M}\,\frac{\sin[(k-\ell)\frac{4\pi}{N}\frac{L}{\lambda}]}{(k-\ell)\frac{4\pi}{N}\frac{L}{\lambda}} \qquad \text{(A-14)}$$

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